ON TWO 10TH ORDER MOCK THETA IDENTITIES

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Abstract. We give short proofs of conjectural identities due to Gordon and McIntosh involving two 10th order mock theta functions.

1. Introduction

Two of the "10th order" mock theta functions found on page 9 of Ramanujan's lost notebook [8] are

$$X(q) := \sum_{n\geq 0} \frac{(-1)^n q^{n^2}}{(-q;q)_{2n}}$$

and

$$\chi(q) := \sum_{n\geq 0} \frac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}}.$$

Here and throughout, we use the standard q-hypergeometric notation

$$(a)_n = (a;q)_n = \prod_{k=1}^{n} (1 - aq^{k-1}),$$

valid for $n \in \mathbb{N} \cup \{\infty\}$. In the spirit of the celebrated "5th order" mock theta conjectures [1, 5], Gordon and McIntosh have recently conjectured the following identities for $X(q)$ and $\chi(q)$ (see (5.18) in [4] or [7]):

$$\begin{aligned} X(-q^2) &= -2qg_2(q,q^{20}) + 2q^5 g_2(q^9,q^{20}) \\ &\quad + \frac{(q^4;q^4)_\infty^2 \big(j(-q^2,q^{20})^2 j(q^{12},q^{40}) + 2q(q^{40};q^{40})_\infty^3\big)}{(q^2;q^2)_\infty (q^{20};q^{20})_\infty (q^{40};q^{40})_\infty j(q^8,q^{40})} \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} \chi(-q^2) &= -2q^3 g_2(q^3,q^{20}) - 2q^5 g_2(q^7,q^{20}) \\ &\quad + \frac{q^2(q^4;q^4)_\infty^2 \big(2q(q^{40};q^{40})_\infty^3 - j(-q^6,q^{20})^2 j(q^4,q^{40})\big)}{(q^2;q^2)_\infty (q^{20};q^{20})_\infty (q^{40};q^{40})_\infty j(q^{16},q^{40})}, \end{aligned} \tag{1.2}$$

where $g_2(x,q)$ is a so-called "universal mock theta function"

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$$g_2(x,q) := \frac{1}{j(q,q^2)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{n(n+1)}}{1-xq^n}$$

and $j(x,q) := (x)_\infty (q/x)_\infty (q)_\infty$.

As stated in [7], identities (1.1) and (1.2) were discovered by using computer algebra and "a rigorous proof has yet to be worked out". It is now well-known that such mock theta conjectures can be reduced to a finite (but possibly formidable) computation by using the theory of harmonic weak Maass forms (see [3] for an example of such an argument). Instead of following this approach, we observe that these two identities follow easily upon appealing to results of Choi [2] which express $X(q)$ and $\chi(q)$ in terms of Appell-Lerch sums and modular forms, applying properties of Appell-Lerch sums and verifying a simple modular form identity. Since all classical mock theta functions can be written in terms of Appell-Lerch sums (see Section 4 of [6]), one can also easily prove identities similar to (1.1) and (1.2) for 2nd, 3rd, 6th and 8th order mock theta functions, see (5.2), (3.12), (5.10) and the top of page 125 in [4]. The details are left to the interested reader. Our main result is the following.

Theorem 1.1. *Identities (1.1) and (1.2) are true.*

2. Preliminaries

We recall some required facts which are conveniently given in [6]. For x, $z \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ with neither z nor xz an integral power of q, define the Appell-Lerch sums

$$m(x,q,z) := \frac{1}{j(z,q)} \sum_{r\in\mathbb{Z}} \frac{(-1)^r q^{\binom{r}{2}} z^r}{1-q^{r-1}xz}.$$

Also, if x^2 is neither zero nor an integral power of q^2, then define

$$k(x,q) := \frac{1}{xj(-q,q^4)} \sum_{n\in\mathbb{Z}} \frac{q^{n(2n+1)}}{1-q^{2n}x^2}.$$

Following [6], we use the term "generic" to mean that the parameters do not cause poles in the the Appell-Lerch sums or in the quotients of theta functions. For generic x, z, z_0 and $z_1 \in \mathbb{C}^*$, the sums $m(x,q,z)$ satisfy (see (2.2b) of Proposition 2.1, Theorem 2.3, Corollary 2.7, Proposition 3.4 and Proposition 3.6 in [6])

$$m(x,q,z) = x^{-1} m(x^{-1},q,z^{-1}), \tag{2.1}$$

$$m(x,q,z_1) = m(x,q,z_0) + \Delta(x,q,z_1,z_0), \tag{2.2}$$

$$m(x,q,z) = m(-qx^2,q^4,z^4) - q^{-1}xm(-q^{-1}x^2,q^4,z^4) - \Lambda(x,q,z), \tag{2.3}$$

$$g_2(x,q) = -x^{-1} m(x^{-2}q,q^2,x) \tag{2.4}$$

and

$$xk(x,q) = m(-x^2,q,x^{-2}) + \frac{J_1^4}{2J_2^2 j(x^2,q)}. \tag{2.5}$$

Here

$$\Delta(x,q,z_1,z_0) := \frac{z_0 J_1^3 j(z_1/z_0,q) j(xz_0z_1,q)}{j(z_0,q)j(z_1,q)j(xz_0,q)j(xz_1,q)},$$

$$\Lambda(x,q,z) := \frac{J_2J_4 j(-xz^2,q)j(-xz^3,q)}{xj(xz,q)j(z^4,q^4)j(-qx^2z^4,q^2)}$$

and $J_m := J_{m,3m}$ with $J_{a,m} = J_{a,m}(q) := j(q^a,q^m)$.

3. Proof of Theorem 1.1

Proof of Theorem 1.1. As observed in [6], after replacing q with $-q^2$ in $X(q)$ on page 183 of [2] and using (2.5) followed by (2.2), we have

$$\begin{aligned} X(-q^2) &= -2q^2k(-q^2,-q^{10}) - \frac{J_5(-q^2)J_{10}(-q^2)J_{2,5}(-q^2)}{J_{2,10}(-q^2)J_{1,5}(-q^2)} \\ &= 2m(-q^4,-q^{10},q^8) - \frac{J_{3,10}(-q^2)J_{5,10}(-q^2)}{J_{1,5}(-q^2)}. \end{aligned} \tag{3.1}$$

Now, taking $x = -q^4$, $q \to -q^{10}$ and $z = q^8$ in (2.3) and simplifying via (2.1) yields

$$m(-q^4,-q^{10},q^8) = m(q^{18},q^{40},q^{32}) - q^{-4}m(q^2,q^{40},q^{-32}). \tag{3.2}$$

Here, $\Lambda(-q^4,-q^{10},q^8) = 0$. By (2.2), we have

$$m(q^{18},q^{40},q^{32}) = m(q^{18},q^{40},q) + \Delta(q^{18},q^{40},q^{32},q) \tag{3.3}$$

and

$$m(q^2,q^{40},q^{-32}) = m(q^2,q^{40},q^9) + \Delta(q^2,q^{40},q^{-32},q^9). \tag{3.4}$$

Comparing (1.1) with (2.4) and (3.1)–(3.4), it now suffices to prove the identity

$$\begin{aligned} &\frac{(q^4;q^4)_\infty^2\big(j(-q^2,q^{20})^2 j(q^{12},q^{40}) + 2q(q^{40};q^{40})_\infty^3\big)}{(q^2;q^2)_\infty(q^{20};q^{20})_\infty(q^{40};q^{40})_\infty j(q^8,q^{40})} \\ &= -\frac{J_{3,10}(-q^2)J_{5,10}(-q^2)}{J_{1,5}(-q^2)} + 2\Delta(q^{18},q^{40},q^{32},q) - 2q^{-4}\Delta(q^2,q^{40},q^{-32},q^9). \end{aligned}$$

This identity has been verified using Garvan's MAPLE program, see

`http://www.math.ufl.edu/~fgarvan/qmaple/theta-supplement/`

This proves (1.1).

After replacing q with $-q^2$ in $\chi(q)$ on page 183 of [2] and again using (2.5) followed by (2.2), we have

$$\begin{aligned}\chi(-q^2) &= 2 - 2q^4 k(q^4, -q^{10}) - q^2 \frac{J_5(-q^2)J_{10}(-q^2)J_{1,5}(-q^2)}{J_{4,10}(-q^2)J_{2,5}(-q^2)} \\ &= 2m(q^2, -q^{10}, q^4) - \frac{q^2 J_{1,10}(-q^2)J_{5,10}(-q^2)}{J_{2,5}(-q^2)}.\end{aligned} \tag{3.5}$$

Now, taking $x = q^2$, $q \to -q^{10}$ and $z = q^4$ in (2.3) and simplifying via (2.1) yields

$$m(q^2, -q^{10}, q^4) = m(q^{14}, q^{40}, q^{16}) + q^{-2} m(q^6, q^{40}, q^{-16}). \tag{3.6}$$

Here, $\Lambda(q^2, -q^{10}, q^4) = 0$. By (2.2), we have

$$m(q^{14}, q^{40}, q^{16}) = m(q^{14}, q^{40}, q^3) + \Delta(q^{14}, q^{40}, q^{16}, q^3) \tag{3.7}$$

and

$$m(q^6, q^{40}, q^{-16}) = m(q^6, q^{40}, q^7) + \Delta(q^6, q^{40}, q^{-16}, q^7). \tag{3.8}$$

Comparing (1.2) with (2.4) and (3.5)–(3.8), it now suffices to prove the identity

$$\begin{aligned}&\frac{(q^4;q^4)_\infty^2 \big(2q(q^{40};q^{40})_\infty^3 - j(-q^6, q^{20})^2 j(q^4, q^{40})\big)}{(q^2;q^2)_\infty (q^{20};q^{20})_\infty (q^{40};q^{40})_\infty j(q^{16}, q^{40})} \\ &= -q^2 \frac{J_{1,10}(-q^2)J_{5,10}(-q^2)}{J_{2,5}(-q^2)} + 2\Delta(q^{14}, q^{40}, q^{16}, q^3) + 2q^{-2}\Delta(q^6, q^{40}, q^{-16}, q^7).\end{aligned}$$

This identity has also been verified using Garvan's MAPLE program. This proves (1.2).

□

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References

[1] G. E. Andrews and F. Garvan, *Ramanujan "lost" notebook. VI. The mock theta conjectures*, Adv. in Math. **73** (1989), no. 2, 242–255.
[2] Y.-S. Choi, *Tenth order mock theta functions in Ramanujan's lost notebook II*, Adv. in Math. **156** (2000), no. 2, 180–285.
[3] A. Folsom, *A short proof of the mock theta conjectures using Maass forms*, Proc. Amer. Math. Soc. **136** (2008), no. 12, 4143–4149.
[4] B. Gordon and R.J. McIntosh, *A survey of classical mock theta functions*, in: Partitions, q-series, and Modular Forms, Developments in Mathematics 2012, vol. 23, 95–144.
[5] D. Hickerson, *A proof of the mock theta conjectures*, Invent. Math. **94** (1988), no. 3, 639–660.
[6] D. Hickerson and E. Mortenson, *Hecke-type double sums, Appell-Lerch sums, and mock theta functions (I)*, preprint available at `http://arxiv.org/abs/1208.1421`
[7] R.J. McIntosh, *The H and K family of mock theta functions*, Canad. J. Math. **64** (2012), no. 4, 935–960.
[8] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa Publishing House, New Delhi, 1988.

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