A q-MULTISUM IDENTITY ARISING FROM FINITE CHAIN RING PROBABILITIES

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Abstract. In this note, we prove a general identity between a q-multisum $B_N(q)$ and a sum of N^2 products of quotients of theta functions. The q-multisum $B_N(q)$ recently arose in the computation of a probability involving modules over finite chain rings.

1. Introduction

Probabilistic proofs of classical q-series identities constitute an intriguing part of the literature in combinatorics. A prominent example of this perspective concerns the Andrews-Gordon identities [1, 10] which state for $1 \leq i \leq k$ and $k \geq 2$

$$\sum_{n_1,\dots,n_{k-1}\geq 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}}}{(q)_{n_1}\cdots(q)_{n_k}} = \prod_{\substack{s=1 \\ s\not\equiv 0,\pm i \pmod{2k+1}}}^{\infty} \frac{1}{1-q^s}, \tag{1.1}$$

where $N_j = n_j + \cdots + n_{k-1}$. Here and throughout, we use the standard q-hypergeometric (or "q-Pochhammer symbol") notation

$$(a)_n = (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k),$$

valid for $n \in \mathbb{N} \cup \{\infty\}$. In [9], Fulman uses a Markov chain on the nonnegative integers to prove the extreme cases $i = 1$ and $i = k$ of (1.1). Chapman [3] cleverly extends Fulman's methods to prove (1.1) in full generality. In [4], Cohen explicitly computes probability laws of p^ℓ-ranks of finite abelian groups to give a group-theoretic proof of (1.1). For a generalization of this computation, see [5]. In this note, we are interested in a recent probability computation with a ring-theoretic flavor as it leads to an expression similar to the left-hand side of (1.1).

Our focus is on finite chain rings, a notion we now briefly recall (for further details, see Section 2 in both [2] and [12]). A ring is called a left (resp. right) chain ring if its lattice of left (resp. right) ideals forms a chain. Any finite chain ring is a local ring, i.e., it has a unique maximal ideal which coincides with its radical. Let $\mathcal{R}$ be a finite chain ring with radical $\mathcal{N}$, q be the order of the residue field $\mathcal{R}/\mathcal{N}$ and N be the index of nilpotency of $\mathcal{N}$. Recently, the authors of [2] expressed the density $\psi(n,k,q,N)$ of free submodules $\mathcal{M}$ of $\mathcal{R}^n$ (over $\mathcal{R}$) of length $k := \log_q(|\mathcal{M}|)$ as $n \to \infty$ as the reciprocal of the q-multisum (replacing $1/q$ in their notation with q)

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$$B_N(q) := \sum_{\substack{K_2,\dots,K_N \geq 0 \\ N | K_2+\cdots+K_N}} \frac{q^{K_2^2+\cdots+K_N^2-(K_2+\cdots+K_N)^2/N}}{(q)_{k_2}\cdots(q)_{k_N}}, \tag{1.2}$$

where $N \geq 2$ is an integer and $K_i = \sum_{j=2}^{i} k_j$. Upper and lower bounds for $B_N(q)$ are obtained and then used to show (under suitable conditions) that $\psi(n,k,q,N)$ is at least $1-\epsilon$ where $0 < \epsilon < 1$ (see Theorems 6 and 8, respectively, in [2]). Moreover, we have

$$B_2(q) = \prod_{\substack{s=1 \\ s \equiv \pm 2, \pm 3, \pm 4, \pm 5 \pmod{16}}}^{\infty} \frac{1}{1-q^s}, \tag{1.3}$$

which is (S.83) in [15]. In view of (1.1) and (1.3), the authors in [2] posed the following (slightly rewritten) problem.

Problem 1.1. *Determine whether $B_N(q)$ can be expressed as a product of q-Pochhammer symbols.*

The purpose of this note is to solve Problem 1.1. It turns out that the solution is slightly more involved than either (1.1) or (1.3), namely $B_N(q)$ is a sum of N^2 products of quotients of theta functions (but not a single product of q-Pochhammer symbols, for general N). Before stating our main result, we recall some further standard notation:

$$\begin{aligned} j(x;q) &:= (x)_\infty (q/x)_\infty (q)_\infty, \\ j(x_1,x_2,\dots,x_n;q) &:= j(x_1;q)j(x_2;q)\cdots j(x_n;q), \\ J_{a,m} &:= j(q^a;q^m), \\ \overline{J}_{a,m} &:= j(-q^a;q^m), \\ J_m &:= J_{m,3m} = (q^m;q^m)_\infty. \end{aligned}$$

Note that these quantities are products of q-Pochhammer symbols. Our main result is now the following.

Theorem 1.2. *For all $N \geq 2$, we have*

$$\begin{aligned} B_N(q) = \frac{1}{(q)_\infty^2 \overline{J}_{0,N(N+2)}} \sum_{r=0}^{N-1}\sum_{s=0}^{N-1} & \frac{(-1)^{r+s+1} q^{\binom{r}{2}+\binom{s+1}{2}+r(s+1)(N+1)+r+s+1} J^3_{N^2(N+2)}}{j((-1)^N q^{N(N+2)r+N(N+3)/2}; q^{N^2(N+2)})} \\ & \times \frac{j(-q^{N(s-r)};q^{N^2}) j(q^{N(N+2)(r+s)+N(N+3)}; q^{N^2(N+2)})}{j((-1)^N q^{N(N+2)s+N(N+3)/2}; q^{N^2(N+2)})}. \end{aligned} \tag{1.4}$$

Formula (1.4) is of interest for at least two reasons. First, Andrews-Gordon type q-multisums akin to (1.1) are typically evaluated as single infinite products using q-series methods such as Bailey pairs, the triple product identity or the quintuple product identity. Instances of q-multisums which evaluate to sums of infinite products seem to be less well-studied and thus certainly require further attention. For pertinent work involving character formulas of irreducible highest weight modules of Kac-Moody algebras of affine type, see [6, 7]. Second, in order to compute asymptotics or find congruences for the coefficients of q-multisums, one would ideally prefer a single infinite product expression. In lieu of this situation, sums of infinite products

$N \setminus 1/q$	2	3	5	7	11
2	0.59546	0.84191	0.95049	0.97627	0.99092
3	0.47084	0.79666	0.94102	0.97295	0.99010
4	0.42109	0.78230	0.93915	0.97248	0.99002
5	0.39877	0.77759	0.93877	0.97241	0.99002
6	0.38819	0.77603	0.93870	0.97240	0.99002
7	0.38304	0.77551	0.93868	0.97240	0.99002
8	0.38050	0.77533	0.93868	0.97240	0.99002
9	0.37924	0.77528	0.93868	0.97240	0.99002
10	0.37861	0.77526	0.93868	0.97240	0.99002
100	0.37798	0.77525	0.93868	0.97240	0.99002
$(q)_\infty$	0.28879	0.56013	0.76033	0.83680	0.90083

TABLE 1. Values of $B_N(q)$

are often still helpful. Indeed, contrarily to (1.2) which requires computing a $(N-1)$-fold sum, (1.4) only involves a double sum. As a comparison with Table 1 in [2], we explicitly compute $B_N(q)$ for $2 \le N \le 10$ and $N = 100$ and $1/q = 2, 3, 5, 7, 11$ to five decimals with Maple using (1.4). Table 1 above suggests that when $q \to 0$, the limiting value of $B_N(q)$ is 1. This statement is confirmed in [2, Corollary 10, (1)].

The paper is organized as follows. In Section 2, we recall one of the main results from [17], then prove Theorem 1.2. In Section 3, we make some concluding remarks.

2. Proof of Theorem 1.2

Before the proof of Theorem 1.2, we need to recall some background from the important work of Hickerson and Mortenson [17]. First, we employ the Hecke-type series

$$f_{a,b,c}(x,y,q) := \Big(\sum_{r,s\ge 0} - \sum_{r,s<0}\Big)(-1)^{r+s}x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}. \tag{2.1}$$

Next, consider the Appell-Lerch series

$$m(x,q,z) := \frac{1}{j(z;q)}\sum_{r\in\mathbb{Z}}\frac{(-1)^r q^{\binom{r}{2}} z^r}{1-q^{r-1}xz}, \tag{2.2}$$

where $x,\, z \in \mathbb{C}^* := \mathbb{C}\setminus\{0\}$ with neither z nor xz an integral power of q in order to avoid poles. One of the main results in [17] expresses (2.1) in terms of (2.2). Let

$$\begin{aligned} g_{a,b,c}(x,y,q,z_1,z_0) := &\sum_{t=0}^{a-1}(-y)^t q^{c\binom{t}{2}} j(q^{bt}x;q^a)\, m\Big(-q^{a\binom{b+1}{2}-c\binom{a+1}{2}-t(b^2-ac)}\frac{(-y)^a}{(-x)^b}, q^{a(b^2-ac)}, z_0\Big)\\ &+\sum_{t=0}^{c-1}(-x)^t q^{a\binom{t}{2}} j(q^{bt}y;q^c)\, m\Big(-q^{c\binom{b+1}{2}-a\binom{c+1}{2}-t(b^2-ac)}\frac{(-x)^c}{(-y)^b}, q^{c(b^2-ac)}, z_1\Big). \end{aligned} \tag{2.3}$$

Following [17], we use the term "generic" to mean that the parameters do not cause poles in the Appell-Lerch sums or in the quotients of theta functions.

Theorem 2.1 ([17], Theorem 1.3)**.** *Let n and p be positive integers with $(n,p)=1$. For generic $x,\ y \in \mathbb{C}^*$,*

$$f_{n,n+p,n}(x,y,q) = g_{n,n+p,n}(x,y,q,-1,-1) + \frac{1}{\overline{J}_{0,np(2n+p)}}\theta_{n,p}(x,y,q),$$

where

$$\theta_{n,p}(x,y,q) := \sum_{r^*=0}^{p-1}\sum_{s^*=0}^{p-1} q^{n\binom{r-(n-1)/2}{2}+(n+p)(r-(n-1)/2)(s+(n+1)/2)+n\binom{s+(n+1)/2}{2}}(-x)^{r-(n-1)/2}$$
$$\times \frac{(-y)^{s+(n+1)/2}J^3_{p^2(2n+p)}j(-q^{np(s-r)}\frac{x^n}{y^n};q^{np^2})j(q^{p(2n+p)(r+s)+p(n+p)}(xy)^p;q^{p^2(2n+p)})}{j(q^{p(2n+p)r+p(n+p)/2}\frac{(-y)^{n+p}}{(-x)^n},q^{p(2n+p)s+p(n+p)/2}\frac{(-x)^{n+p}}{(-y)^n};q^{p^2(2n+p)})}.$$

Here, $r := r^* + \{(n-1)/2\}$ and $s := s^* + \{(n-1)/2\}$ with $0 \leq \{\alpha\} < 1$ denoting the fractional part of α.

We can now prove Theorem 1.2.

Proof of Theorem 1.2. The first step is to recognize $B_N(q)$ in a different context. For $N \geq 1$, consider the string function of level N of the affine Lie algebra $A_1^{(1)}$ (e.g., see [14, 19])

$$\mathcal{C}^N_{m,\ell}(q) = \frac{q^{\frac{m^2-\ell^2}{4N}}}{(q)_\infty} \sum_{\substack{\mathbf{n}\in\mathbb{Z}^{N-1}_{\geq 0}\\ \frac{m+\ell}{2N}+(C^{-1}\mathbf{n})_1\in\mathbb{Z}}} \frac{q^{\mathbf{n}C^{-1}(\mathbf{n}-\mathbf{e}_\ell)^T}}{(q)_{n_1}\cdots(q)_{n_{N-1}}}, \tag{2.4}$$

where $\mathbf{n} = (n_1,\dots,n_{N-1})$, $\mathbf{e}_i$ is the i-th standard unit vector in $\mathbb{Z}^{N-1}$ (with $\mathbf{e}_0 = \mathbf{e}_N = 0$), C is the A_{N-1} Cartan matrix whose inverse C^{-1} is given by

$$(C^{-1})_{i,j} = \min(i,j) - \frac{ij}{N},$$

and $(C^{-1}\mathbf{n})_1$ is the first entry in the vector $C^{-1}\mathbf{n}$. A straightforward computation (see the proof of Theorem 5 in [2]) yields

$$B_N(q) = \sum_{\substack{\mathbf{n}\in\mathbb{Z}^{N-1}_{\geq 0}\\ (C^{-1}\mathbf{n})_1\in\mathbb{Z}}} \frac{q^{\mathbf{n}C^{-1}\mathbf{n}^T}}{(q)_{n_1}\cdots(q)_{n_{N-1}}}. \tag{2.5}$$

Comparing (2.4) when $\ell = 0$ and m is divisible by $2N$ with (2.5), we have for all $N \geq 2$,

$$B_N(q) = q^{\frac{-m^2}{4N}}(q)_\infty \mathcal{C}^N_{m,0}(q). \tag{2.6}$$

Next, by Example 1.3 on page 386 of [17], we have

$$\mathcal{C}^N_{m,0}(q) = \frac{1}{(q)^3_\infty} f_{1,N+1,1}(q^{1+m/2},q^{1-m/2},q).$$

Thus from (2.6), we deduce that for all $N \geq 2$ and m divisible by $2N$,

$$B_N(q) = \frac{q^{\frac{-m^2}{4N}}}{(q)_\infty^2} f_{1,N+1,1}(q^{1+m/2}, q^{1-m/2}, q). \tag{2.7}$$

By Theorem 2.1, we have

$$\begin{aligned} f_{1,N+1,1}(q^{1+m/2}, q^{1-m/2}, q) &= g_{1,N+1,1}(q^{1+m/2}, q^{1-m/2}, q, -1, -1) \\ &\quad + \frac{1}{\overline{J}_{0,N(N+2)}} \theta_{1,N}(q^{1+m/2}, q^{1-m/2}, q). \end{aligned}$$

Now, observe that

$$g_{1,N+1,1}(q^{1+m/2}, q^{1-m/2}, q, -1, -1) = 0$$

as there are no poles in the Appell-Lerch series

$$m(q^{N(N+1)/2+m(N+2)/2}, q^{N(N+2)}, -1)$$

and

$$m(q^{N(N+1)/2-m(N+2)/2}, q^{N(N+2)}, -1)$$

(indeed, this is true whenever $m(N+2)/2 \not\equiv \pm N(N+1)/2 \pmod{N(N+2)}$, which is always the case when $m \equiv 0 \pmod{2N}$) and $j(q^{1+m/2}; q) = j(q^{1-m/2}; q) = 0$. Thus,

$$B_N(q) = \frac{q^{\frac{-m^2}{4N}}}{(q)_\infty^2 \overline{J}_{0,N(N+2)}} \theta_{1,N}(q^{1+m/2}, q^{1-m/2}, q).$$

We now take $m = 0$. The factor $q^{\frac{-m^2}{4N}}$ disappears and $\theta_{1,N}(q, q, q)$ is given as in (1.4). This proves the result.

$\square$

3. Concluding remarks

There are several avenues for further study. First, Table 1 suggests that as $N \to \infty$, the limiting value of $B_N(q)$ is strictly larger than $(q)_\infty$. This is a stronger statement than [2, Corollary 10, (2)]. Thus, it would be desirable to compute both asymptotics for $B_N(q)$ and the correct limiting value of $\psi(n, k, q, N)$ as $N \to \infty$. Second, for $N = 2$, 3 and 4, one can reduce the number of products of quotients of theta functions occurring in Theorem 1.2 by first invoking Theorems 1.9–1.11 in [17], then performing routine (yet possibly involved) simplifications [8]. In these cases, we require that $m \equiv 0 \pmod{2N}$, $m \not\equiv 0 \pmod{N(N+2)}$ and, if m is odd, $m \not\equiv \pm(N+1) \pmod{2(N+2)}$. For example, one can recover (1.3) in this manner. The details are left to the interested reader. Third, given that (2.6) is a key step in the proof of Theorem 1.2, it is natural to wonder if string functions which generalize (2.4) (see [11, 13]) can also be realized in terms of computing an appropriate probability. For recent related works on string functions, see [16, 18]. Finally, can Theorem 1.2 be understood via Markov chains, group theory or, possibly, Hall-Littlewood functions [20]?

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References

[1] G.E. Andrews, *An analytic generalization of the Rogers-Ramanujan identities for odd moduli*, Proc. Nat. Acad. Sci. U.S.A. **74** (1974), 4082–4085.

[2] E. Byrne, A-L Horlemann, K. Khathuria and V. Weger, *Density of free modules over finite chain rings*, preprint available at `https://arxiv.org/abs/2106.09403`

[3] R. Chapman, *A probabilistic proof of the Andrews-Gordon identities*, Discrete Math. **290** (2005), no. 1, 79–84.

[4] H. Cohen, *On the p^k-rank of finite abelian groups and Andrews' generalization of the Rogers-Ramanujan identities*, Nederl. Akad. Wetensch. Indag. Math. **47** (1985), no. 4, 377–383.

[5] C. Delaunay, *Averages of groups involving p^ℓ-rank and combinatorial identities*, J. Number Theory 131 (2011), no. 3, 536–551.

[6] J. Dousse, I. Konan, *Generalisations of Capparelli's and Primc's identities, I: coloured Frobenius partitions and combinatorial proofs*, preprint available at `https://arxiv.org/abs/1911.13191`

[7] J. Dousse, I. Konan, *Multi-grounded partitions and character formulas*, Adv. Math. **400** (2022), 108275, 41 pp.

[8] J. Frye, F.G. Garvan, *Automatic proof of theta-function identities*, Elliptic integrals, elliptic functions and modular forms in quantum field theory, 195–258, Texts Monogr. Symbol. Comput., Springer, Cham, 2019.

[9] J. Fulman, *A probabilistic proof of the Rogers-Ramanujan identities*, Bull. London Math. Soc. **33** (2001), no. 4, 397–407.

[10] B. Gordon, *A combinatorial generalization of the Rogers-Ramanujan identities*, Amer. J. Math. **83** (1961), 393–399.

[11] G. Hatayama, A.N. Kirillov, A.. Kuniba, M. Okado, T. Takagi and Y. Yamada, *Character formulae of $\widehat{sl_n}$-modules and inhomogeneous paths*, Nuclear Phys. B **536** (1999), no. 3, 575–616.

[12] T. Honold, I. Landjev, *Linear codes over finite chain rings*, Electron. J. Combin. **7** (2000), Research Paper 11, 22pp.

[13] A. Kuniba, T. Nakanishi and J. Suzuki, *Characters in conformal field theories from thermodynamic Bethe ansatz*, Modern Phys. Lett. A **8** (1993), no. 18, 1649–1659.

[14] J. Lepowsky, M. Primc, *Structure of the standard modules for the affine Lie algebra $A_1^{(1)}$*, Contemporary Mathematics, **46**. American Mathematical Society, Providence, RI, 1985.

[15] J. McLaughlin, A. Sills and P. Zimmer, *Rogers-Ramanujan-Slater type identities*, Electron. J. Combin. **15** (1998), Dynamic Surveys 15, 59pp.

[16] E. Mortenson, *On Hecke-type double-sums and general string functions for the affine Lie algebra $A_1^{(1)}$*, available at `https://arxiv.org/abs/2110.02615`

[17] E. Mortenson, D. Hickerson, *Hecke-type double sums, Appell-Lerch sums, and mock theta functions, I*, Proc. Lond. Math. Soc. (3) **109** (2014), no. 2, 382–422.

[18] E. Mortenson, O. Postnova and D. Solovyev, *On string functions and double-sum formulas*, available at `https://arxiv.org/abs/2107.06225`

[19] A. Schilling, O. Warnaar, *Conjugate Bailey pairs: from configuration sums to fractional-level string functions to Bailey's lemma*, Recent developments in infinite-dimensional Lie algebras and conformal field theory (Charlottesville, VA, 2000), 227–255, Contemp. Math., **297**, Amer. Math. Soc., Providence, RI, 2002.

[20] J. Stembridge, *Hall-Littlewood functions, plane partitions, and the Rogers-Ramanujan identities*, Trans. Amer. Math. Soc. 319 (1990), no. 2, 469–498.

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