A NOTE ON 4-RANK DENSITIES

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ABSTRACT. For certain real quadratic number fields, we prove density results concerning 4-ranks of tame kernels. We also discuss a relationship between 4-ranks of tame kernels and 4-class ranks of narrow ideal class groups. Additionally, we give a product formula for a local Hilbert symbol.

1. INTRODUCTION

Let F be a real quadratic number field and $\mathcal{O}_F$ its ring of integers. In [4], the authors gave an algorithm for computing the 4-rank of the tame kernel $K_2(\mathcal{O}_F)$. The idea of the algorithm is to consider matrices with Hilbert symbols as entries and compute matrix ranks over $\mathbb{F}_2$. Recently, the author used these matrices to obtain "density results" concerning the 4-rank of tame kernels, see [6], [7].

In this note, we consider the 4-rank of $K_2(\mathcal{O})$ for the real quadratic number fields $\mathbb{Q}(\sqrt{p_1 p_2 p_3})$ for primes $p_1 \equiv p_2 \equiv p_3 \equiv 1 \bmod 8$. We will see that

$$\text{4-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1 p_2 p_3})}) = 0, 1, 2, \text{ or } 3.$$

For squarefree, odd integers d, consider the set

$$X = \{d : d = p_1 p_2 p_3, \ p_i \equiv 1 \bmod 8\}$$

for distinct primes p_i.

Using GP/PARI [1], we computed the following: For $50881 \leq d < 2 \times 10^7$, there are 7257 d's in X. Among them, there are 2121 d's (29.23%) yielding 4-rank 0, 3977 d's (54.80%) yielding 4-rank 1, 1086 d's (14.96%) yielding 4-rank 2, and 73 d's (1.01%) yielding 4-rank 3. In fact, we prove

Theorem 1.1. *For the fields* $\mathbb{Q}(\sqrt{p_1 p_2 p_3})$, *4-rank 0, 1, 2, and 3 appear with natural density* $\frac{1}{4}$, $\frac{17}{32}$, $\frac{13}{64}$, *and* $\frac{1}{64}$ *respectively in X.*

In the appendix we point out a beautiful result which may not be well known. It is a product formula from [11] for a certain local Hilbert symbol. This product formula both simplifies numerical computations and is a generalization of Propositions 4.6 and 4.4 in [2] and [7], respectively.

2. MATRICES

Hurrelbrink and Kolster [4] generalize Qin's approach in [8], [9] and obtain 4-rank results by computing $\mathbb{F}_2$-ranks of certain matrices of local Hilbert symbols. Specifically, let $F = \mathbb{Q}(\sqrt{d})$, $d > 1$ and squarefree. Let $p_1, p_2, \ldots, p_t$ denote the

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odd primes dividing d. Recall 2 is a norm from F if and only if all p_i's are $\equiv \pm 1 \bmod 8$. If so, then d is a norm from $\mathbb{Q}(\sqrt{2})$, thus

$$d = u^2 - 2w^2$$

for $u, w \in \mathbb{Z}$. Now consider the matrix:

$$M_{F/\mathbb{Q}} = \begin{pmatrix} (-d, p_1)_2 & (-d, p_1)_{p_1} & \cdots & (-d, p_1)_{p_t} \\ (-d, p_2)_2 & (-d, p_2)_{p_1} & \cdots & (-d, p_2)_{p_t} \\ \vdots & \vdots & & \vdots \\ (-d, p_{t-1})_2 & (-d, p_{t-1})_{p_1} & \cdots & (-d, p_{t-1})_{p_t} \\ (-d, v)_2 & (-d, v)_{p_1} & \cdots & (-d, v)_{p_t} \\ (d, -1)_2 & (d, -1)_{p_1} & \cdots & (d, -1)_{p_t} \end{pmatrix}.$$

If 2 is not a norm from F, set $v = 2$. Otherwise, set $v = u + w$. Replacing the 1's by 0's and the -1's by 1's, we calculate the matrix rank over $\mathbb{F}_2$. From [4],

Lemma 2.1. *Let $F = \mathbb{Q}(\sqrt{d})$, $d > 0$ and squarefree. Then*

$$\text{4-rank } K_2(\mathcal{O}_F) = t - \text{rk } (M_{F/\mathbb{Q}}) + a' - a$$

where

$$a = \begin{cases} 0 & \text{if 2 is a norm from F} \\ 1 & \text{otherwise} \end{cases}$$

and

$$a' = \begin{cases} 0 & \text{if both -1 and 2 are norms from F} \\ 1 & \text{if exactly one of -1 or 2 is a norm from F} \\ 2 & \text{if none of -1 or 2 are norms from F.} \end{cases}$$

Recall that our case is $\mathbb{Q}(\sqrt{p_1 p_2 p_3})$ for primes $p_1 \equiv p_2 \equiv p_3 \equiv 1 \bmod 8$. In this case $a = a'$ and we may delete the last row of $M_{F/\mathbb{Q}}$ without changing its rank (see discussions preceding Proposition 5.13 and Lemma 5.14 in [4]). Also note that v is an p_1-adic unit and hence

$$(-p_1 p_2 p_3, v)_{p_1} = (p_1, v)_{p_1} = \left(\frac{v}{p_1}\right).$$

Similarly, $(-p_1 p_2 p_3, v)_{p_2} = \left(\frac{v}{p_2}\right)$ and $(-p_1 p_2 p_3, v)_{p_3} = \left(\frac{v}{p_3}\right)$. From Lemma 2.1 we have

$$\text{4-rank } K_2(\mathcal{O}_F) = 3 - \text{rk } (M_{F/\mathbb{Q}})$$

and the matrix $M_{F/\mathbb{Q}}$ is of the form

$$\begin{pmatrix} 1 & \left(\frac{p_2}{p_1}\right)\left(\frac{p_3}{p_1}\right) & \left(\frac{p_1}{p_2}\right) & \left(\frac{p_1}{p_3}\right) \\ 1 & \left(\frac{p_2}{p_1}\right) & \left(\frac{p_1}{p_2}\right)\left(\frac{p_3}{p_2}\right) & \left(\frac{p_2}{p_3}\right) \\ (-d, u+w)_2 & \left(\frac{v}{p_1}\right) & \left(\frac{v}{p_2}\right) & \left(\frac{v}{p_3}\right) \end{pmatrix}.$$

Let us now prove Theorem 1.1.

Proof. The idea in [6] and [7] is to first consider an appropriate normal extension N of $\mathbb{Q}$ and then relate the splitting of the primes p_i in N to their representation by certain quadratic forms. The next step is classifying 4-rank values in terms of values of the symbols $(-d, v)_2$, $\left(\frac{v}{p_i}\right)$. The values of these symbols are then characterized in terms of p_i satisfying the alluded to quadratic forms. We then associate Artin symbols to the primes p_i and apply the Chebotarev density theorem. In what follows, we classify the 4-rank values in terms of the symbols $(-d, v)_2$, $\left(\frac{v}{p_i}\right)$ and in parenthesis give the relevant densities in X obtained by using the above machinery. Let us consider the following four cases (see Table III in [9]).

Case 1: Suppose $\left(\frac{p_2}{p_1}\right) = \left(\frac{p_3}{p_1}\right) = \left(\frac{p_3}{p_2}\right) = 1$. Then we immediately have that

- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1 p_2 p_3})}) = 3 \iff \text{rank}\,(M_{F/\mathbb{Q}}) = 0 \iff (-d, v)_2 = 1$ and $\left(\frac{v}{p_1}\right) = \left(\frac{v}{p_2}\right) = \left(\frac{v}{p_3}\right) = 1$ $(\frac{1}{64})$

- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1 p_2 p_3})}) = 2 \iff \text{rank}\,(M_{F/\mathbb{Q}}) = 1 \iff (-d, v)_2 = -1$ or $(-d, v)_2 = 1$ and $\left(\frac{v}{p_1}\right) = \left(\frac{v}{p_2}\right) = -1$ and $\left(\frac{v}{p_3}\right) = 1$ or $(-d, v)_2 = 1$ and $\left(\frac{v}{p_1}\right) = \left(\frac{v}{p_3}\right) = -1$ and $\left(\frac{v}{p_2}\right) = 1$ or $(-d, v)_2 = 1$ and $\left(\frac{v}{p_2}\right) = \left(\frac{v}{p_3}\right) = -1$ and $\left(\frac{v}{p_1}\right) = 1$ $(\frac{7}{64})$.

Case 2: Suppose $\left(\frac{p_3}{p_2}\right) = \left(\frac{p_3}{p_1}\right) = 1$, $\left(\frac{p_2}{p_1}\right) = -1$. Thus

- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1 p_2 p_3})}) = 2 \iff \text{rank}\,(M_{F/\mathbb{Q}}) = 1 \iff (-d, v)_2 = 1$ and $\left(\frac{v}{p_1}\right) = \left(\frac{v}{p_2}\right) = 1$ or $(-d, v)_2 = 1$ and $\left(\frac{v}{p_1}\right) = \left(\frac{v}{p_2}\right) = -1$ $(\frac{3}{32})$

- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1 p_2 p_3})}) = 1 \iff \text{rank}\,(M_{F/\mathbb{Q}}) = 2 \iff (-d, v)_2 = -1$ or $(-d, v)_2 = 1$ and $\left(\frac{v}{p_1}\right) = \left(\frac{v}{p_3}\right) = -1$ and $\left(\frac{v}{p_2}\right) = 1$ or $(-d, v)_2 = 1$ and $\left(\frac{v}{p_2}\right) = \left(\frac{v}{p_3}\right) = -1$ and $\left(\frac{v}{p_1}\right) = 1$ $(\frac{9}{32})$.

Case 3: Suppose $\left(\frac{p_2}{p_1}\right) = \left(\frac{p_3}{p_1}\right) = -1$, $\left(\frac{p_3}{p_2}\right) = 1$. Thus

- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1 p_2 p_3})}) = 1 \iff \text{rank}\,(M_{F/\mathbb{Q}}) = 2 \iff (-d, v)_2 = 1$ $(\frac{3}{16})$.

- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1 p_2 p_3})}) = 0 \iff \text{rank}\,(M_{F/\mathbb{Q}}) = 3 \iff (-d, v)_2 = -1$ $(\frac{3}{16})$

Case 4: Suppose $\left(\frac{p_2}{p_1}\right) = \left(\frac{p_3}{p_1}\right) = \left(\frac{p_3}{p_2}\right) = -1$. Then

- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1 p_2 p_3})}) = 1 \iff \text{rank}\ (M_{F/\mathbb{Q}}) = 2 \iff (-d, v)_2 = 1$
  $(\frac{1}{16})$

- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1 p_2 p_3})}) = 0 \iff \text{rank}\ (M_{F/\mathbb{Q}}) = 3 \iff (-d, v)_2 = -1$
  $(\frac{1}{16})$.

Thus 4-rank 0, 1, 2, and 3 occur with natural density $\frac{1}{16} + \frac{3}{16} = \frac{1}{4}$, $\frac{1}{16} + \frac{3}{16} + \frac{9}{32} = \frac{17}{32}$, $\frac{3}{32} + \frac{7}{64} = \frac{13}{64}$, and $\frac{1}{64}$.

$\square$

Remark 2.2. The matrices in [4] are related to Rédei matrices which were used in the 1930's to study the structure of narrow ideal class groups. Namely, for $\mathbb{Q}(\sqrt{d})$, we considered the case that all odd primes divisors of d are $\equiv 1 \mod 8$. Thus 2 is a norm from $F = \mathbb{Q}(\sqrt{d})$ and we have the representation

$$d = u^2 - 2w^2.$$

for $u, w \in \mathbb{Z}$. Let $d' = \prod_{i=1}^{t} p_i$. The matrix $M_{F/\mathbb{Q}}$ has the form:

$$\begin{pmatrix} 1 & & & & \\ 1 & & & & \\ \vdots & & \hat{R}_{F/\mathbb{Q}} & & \\ 1 & & & & \\ (-d, v)_2 & (-d, v)_{p_1} & \cdots & (-d, v)_{p_t} \end{pmatrix}.$$

The $(t-1)$ by t matrix $\hat{R}_{F/\mathbb{Q}}$ can be extended, without changing its rank, to a t by t matrix $R_{F/\mathbb{Q}}$ by adding the last row

$$(-d, p_t)_{p_1}, (-d, p_t)_{p_2}, \ldots, (-d, p_t)_{p_t}.$$

$R_{F/\mathbb{Q}}$ is known as the Rédei matrix of the field $F' := \mathbb{Q}(\sqrt{d'})$ (see [5] or [10]). Its rank determines the 4-rank of the narrow ideal class group $C_{F'}^+$ of the field F' by

$$\text{4-rank } C_{F'}^+ = t - 1 - \text{rank}(R_{F/\mathbb{Q}}).$$

Combining this information with Lemma 2.1, we have that if $(-d, u+w)_2 = -1$, then 4-rank $K_2(\mathcal{O}_F) = $ 4-rank $C_{F'}^+$. Using Rédei matrices, Gerth in [3] derived an effective algorithm for computing densities of 4-class ranks of narrow ideal class groups of quadratic number fields. It would be interesting to see if density results concerning 4-class ranks of narrow ideal class groups (coupled with the product formula in the appendix) can be used to obtain asymptotic formulas for 4-rank densities of tame kernels.

3. Appendix: A product formula

Most of the local Hilbert symbols in the martix $M_{F/\mathbb{Q}}$ are calculated directly. Difficulties arise when d is a norm from $\mathbb{Q}(\sqrt{2})$. In this case, we need to calculate the Hilbert symbols $(-d, u + w)_2$ and $(-d, u + w)_{p_k}$. The local symbol at 2 is

calculated using Lemmas 5.3 and 5.4 in [4]. In this appendix we provide a product formula which allows one to calculate $(-d, u+w)_{p_k}$ using 2 factors of d at a time.

Let d be a squarefree integer and assume that all odd prime divisors of d are $\equiv \pm 1 \bmod 8$. Then d is a norm from $F = \mathbb{Q}(\sqrt{2})$ and we have the representation

$$d = u^2 - 2w^2$$

with $u > 0$. Let l be any odd prime dividing d. Note that l does not divide $u + w$ and so

Remark 3.1. $(-d, u + w)_l = (l, u + w)_l = \left(\frac{u+w}{l}\right).$

Recall that any odd prime divisor l of d is $\equiv \pm 1 \bmod 8$. We fix x and y according to the representation:

$$(-1)^{\frac{l-1}{2}} l = N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(x + y\sqrt{2}) = x^2 - 2y^2$$

with $x \equiv 1 \bmod 4$, $x, y > 0$. Observe that mod 8, x is odd. Also we can arrange for $x \equiv 1 \bmod 4$ by multiplying $x + y\sqrt{2}$ by $(1 + \sqrt{2})^2$.

For $l \equiv 1 \bmod 8$, we have $l = x^2 - 2y^2$ and so $\left(\frac{l}{y}\right) = 1$. Thus $\left(\frac{y}{l}\right) = 1$. For $l \equiv 7 \bmod 8$,

$$1 = \left(\frac{-l}{y}\right) = \left(\frac{-1}{y}\right)\left(\frac{l}{y}\right) = (-1)^{\frac{y-1}{2}}(-1)^{\frac{y-1}{2}}\left(\frac{y}{l}\right) = \left(\frac{y}{l}\right).$$

Now let r be an integer not divisible by l which can be represented as a norm from $\mathbb{Q}(\sqrt{2})$. Denote by $\pi_r = s + t\sqrt{2}$ an element such that $N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(\pi_r) = r$ with $s, t > 0$. Now let u_r and w_r be such that

$$u_r + w_r\sqrt{2} = (1 + \sqrt{2})(x + y\sqrt{2})(s + t\sqrt{2}).$$

By the choice of x, y, s, t, we have $u_r > 0$. Note that

$$N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(u_r + w_r\sqrt{2}) = -(-1)^{\frac{l-1}{2}} lr.$$

Now fix $\mathfrak{l} = <x - y\sqrt{2}>$ a prime ideal above l in $\mathbb{Q}(\sqrt{2})$. As l splits in $\mathbb{Q}(\sqrt{2})$, $\mathbb{Z}[\sqrt{2}]/\mathfrak{l} \cong \mathbb{Z}/l\mathbb{Z}$. This allows us to work mod $\mathfrak{l}$ as opposed to mod l. From the above, $u_r + w_r = 2xs + 3tx + 3sy + 4yt$. Modulo $\mathfrak{l}$, we have

$$
\begin{aligned}
u_r + w_r &\equiv 2sy\sqrt{2} + 3ty\sqrt{2} + 3sy + 4yt \\
&\equiv y(3 + 2\sqrt{2})(s + t\sqrt{2}).
\end{aligned}
$$

As $\left(\frac{y}{l}\right) = 1$, $\left(\frac{u_r + w_r}{l}\right) = \left(\frac{y}{l}\right)\left(\frac{\pi_r}{\mathfrak{l}}\right) = \left(\frac{\pi_r}{\mathfrak{l}}\right)$ where

$$\left(\frac{\pi_r}{\mathfrak{l}}\right) = \begin{cases} 1 & \text{if } x^2 \equiv \pi_r \bmod \mathfrak{l} \text{ is solvable} \\ -1 & \text{otherwise.} \end{cases}$$

In the case $r = \prod_i^{t-1} p_i$ where $p_i \equiv \pm 1 \bmod 8$, we obtain for each p_i an element $\pi_i \in \mathbb{Q}(\sqrt{2})$ of norm $(-1)^{\frac{p_i - 1}{2}} p_i$. Let c be the number of primes dividing r which are congruent to 7 modulo 8. Then we have (up to squares of units) $\pi_r = (1 + \sqrt{2})^c \prod_i^{t-1} \pi_i$. This yields

$$\left(\frac{u_r + w_r}{l}\right) = \left(\frac{(1 + \sqrt{2})^c}{\mathfrak{l}}\right) \prod_i^{t-1} \left(\frac{\pi_i}{\mathfrak{l}}\right),$$

and so

$$\left(\tfrac{u_r+w_r}{l}\right) = \left(\tfrac{u_{-1}+w_{-1}}{l}\right)^c \prod_i^{t-1} \left(\tfrac{u_{p_i}+w_{p_i}}{l}\right).$$

As -1 and 2 are also norms from $\mathbb{Q}(\sqrt{2})$, we can include r's having factors -1 or ± 2. Thus for $r = (-1)^n(2)^m \prod_i^{t-1} p_i$ with m, $n = 0$, or 1, and each $p_i \equiv \pm 1 \bmod 8$ and $l \neq p_i$ for any i, we have

Remark 3.2. $\left(\tfrac{u_r+w_r}{l}\right) = \left(\tfrac{u_{-1}+w_{-1}}{l}\right)^{n+c} \left(\tfrac{u_2+w_2}{l}\right)^m \prod_i^{t-1}\left(\tfrac{u_{p_i}+w_{p_i}}{l}\right).$

Setting $r = \tfrac{d}{l}$, we have $-(-1)^{\frac{l-1}{2}}d = N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(u_r + w_r\sqrt{2})$. So for any prime $l \equiv 7 \bmod 8$, we have $N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(u_r + w_r\sqrt{2}) = d = N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(u + w\sqrt{2})$. Then, up to squares, $\left(\tfrac{u_r+w_r}{l}\right) = \left(\tfrac{u+w}{l}\right)$. For prime divisors $l \equiv 1 \bmod 8$, we have $-d = N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(u_r + w_r\sqrt{2})$ and so we include $\left(\tfrac{u_{-1}+w_{-1}}{l}\right)$. To summarize,

Remark 3.3. $(-d, u+w)_l = \begin{cases} \left(\tfrac{u_r+w_r}{l}\right) & \text{if } l \equiv 7 \bmod 8 \\ \left(\tfrac{u_{-1}+w_{-1}}{l}\right)\left(\tfrac{u_r+w_r}{l}\right) & \text{if } l \equiv 1 \bmod 8. \end{cases}$

We may now reduce to the following $d = rl$: $d = -l$, $d = 2l$, and $d = pl$, i.e. calculate the symbols $\left(\tfrac{u_{-1}+w_{-1}}{l}\right)$, $\left(\tfrac{u_2+w_2}{l}\right)$, and $\left(\tfrac{u_p+w_p}{l}\right)$. The first two symbols can be calculated using the following two elementary lemmas.

Lemma 3.4. $\left(\tfrac{u_{-1}+w_{-1}}{l}\right) = 1 \iff (-1)^{\frac{l-1}{2}}l = a^2 - 32b^2$ for some $a,b \in \mathbb{Z}$ with $a \equiv 1 \bmod 4$.

Lemma 3.5. $\left(\tfrac{u_2+w_2}{l}\right) = 1 \iff l \equiv \pm 1 \bmod 16$.

A little care is necessary in computing $\left(\tfrac{u_p+w_p}{l}\right)$. If $\left(\tfrac{(-1)^{\frac{p-1}{2}}p}{l}\right) = 1$, then the symbol $\left(\tfrac{\pi}{l}\right)$ is well defined (see discussion preceeding Proposition 3.5 in [2]) and can be computed using

Lemma 3.6. For $\mathcal{K} = \mathbb{Q}(\sqrt{(-1)^{\frac{p-1}{2}}2p})$ with $p \equiv \pm 1 \bmod 8$ and $h^+(\mathcal{K})$ the narrow class number of $\mathcal{K}$, we have

$\left(\tfrac{\pi}{l}\right) = 1 \iff l^{\frac{h^+(\mathcal{K})}{4}} = n^2 - 2pm^2$ for some $n, m \in \mathbb{Z}$ with $m \not\equiv 0 \bmod l$.

For $\mathcal{K} = \mathbb{Q}(\sqrt{-2p})$ with $p \equiv 7 \bmod 8$, $\left(\tfrac{\pi}{l}\right) = -1 \iff l^{\frac{h^+(\mathcal{K})}{4}} = 2n^2 + pm^2$ for some $n, m \in \mathbb{Z}$ with $m \not\equiv 0 \bmod l$.

For $\mathcal{K} = \mathbb{Q}(\sqrt{2p})$ with $p \equiv 1 \bmod 8$, $\left(\tfrac{\pi}{l}\right) = -1 \iff l^{\frac{h^+(\mathcal{K})}{4}} = pn^2 - 2m^2$ for some $n, m \in \mathbb{Z}$ with $m \not\equiv 0 \bmod l$.

In fact,

Lemma 3.7. If $\left(\tfrac{(-1)^{\frac{p-1}{2}}p}{l}\right) = 1$, then

$$\left(\frac{u_p + w_p}{l}\right) = \begin{cases} \left(\frac{\pi}{l}\right) & for \ p \equiv 1 \bmod 8 \\ \left(\frac{u_{-1} + w_{-1}}{l}\right)\left(\frac{\pi}{l}\right) & for \ p \equiv 7 \bmod 8. \end{cases}$$

The case where $\left(\frac{(-1)^{\frac{p-1}{2}} p}{l}\right) = -1$ can be done by finding u_p and w_p from the presentation

$$N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(u_p + w_p\sqrt{2}) = -(-1)^{\frac{p-1}{2}} pl.$$

Combining Remarks 3.1, 3.2, and 3.3, we have

Theorem 3.8. *For* $d = (-1)^n (2)^m \prod_{i=1}^t p_i$, *with each* $p_i \equiv \pm 1 \bmod 8$, *we have*

$$(-d, u + w)_{p_k} = \left(\frac{u_{-1} + w_{-1}}{p_k}\right)^{n + (-1)^{\frac{p_k + 1}{2}}} \left(\frac{u_2 + w_2}{p_k}\right)^m \prod_{i \neq k}\left(\frac{u_{p_i} + w_{p_i}}{p_k}\right).$$

Example 3.9. Consider the cases $d = \pm pl, \pm 2pl$ with $p \equiv 7 \bmod 8$, $l \equiv 1 \bmod 8$, and $\left(\frac{l}{p}\right) = 1$ (see Proposition 4.6 in [2]). Note that $\left(\frac{\pi}{l}\right)$ is well defined and so Lemma 3.7 is applicable.

For $d = pl$, we have $n = 0$, $m = 0$ and so

$$\begin{aligned} (-d, u + w)_l &= \left(\frac{u_{-1} + w_{-1}}{l}\right)^{-1} \left(\frac{u_2 + w_2}{l}\right)^0 \left(\frac{u_{-1} + w_{-1}}{l}\right)\left(\frac{\pi}{l}\right) \\ &= \left(\frac{\pi}{l}\right). \end{aligned}$$

For $d = 2pl$, we have $n = 0$, $m = 1$. Thus

$$\begin{aligned} (-d, u + w)_l &= \left(\frac{u_{-1} + w_{-1}}{l}\right)^{-1} \left(\frac{u_2 + w_2}{l}\right)^1 \left(\frac{u_{-1} + w_{-1}}{l}\right)\left(\frac{\pi}{l}\right) \\ &= \left(\frac{2 + \sqrt{2}}{l}\right)\left(\frac{\pi}{l}\right). \end{aligned}$$

For $d = -pl$, we have $n = 1$, $m = 0$. This yields

$$\begin{aligned} (-d, u + w)_l &= \left(\frac{u_{-1} + w_{-1}}{l}\right)^0 \left(\frac{u_2 + w_2}{l}\right)^0 \left(\frac{u_{-1} + w_{-1}}{l}\right)\left(\frac{\pi}{l}\right) \\ &= \left(\frac{1 + \sqrt{2}}{l}\right)\left(\frac{\pi}{l}\right). \end{aligned}$$

Finally, for $d = -2pl$, we have $n = 1$, $m = 1$. So

$$\begin{aligned} (-d, u + w)_l &= \left(\frac{u_{-1} + w_{-1}}{l}\right)^0 \left(\frac{u_2 + w_2}{l}\right)^1 \left(\frac{u_{-1} + w_{-1}}{l}\right)\left(\frac{\pi}{l}\right) \\ &= \left(\frac{2 + \sqrt{2}}{l}\right)\left(\frac{1 + \sqrt{2}}{l}\right)\left(\frac{\pi}{l}\right). \end{aligned}$$

Example 3.10. Consider the cases $d = \pm pl$ with $p \equiv l \equiv 1 \bmod 8$, and $\left(\frac{l}{p}\right) = 1$ (see Proposition 4.4 in [7]). Again $\left(\frac{\pi}{l}\right)$ is well defined and so Lemma 3.7 is applicable.

For $d = pl$, we have $n = 0$, $m = 0$, and so

$$
\begin{aligned}
(-d, u+w)_l &= \left(\frac{u_{-1} + w_{-1}}{l}\right)^{-1} \left(\frac{u_2 + w_2}{l}\right)^0 \left(\frac{u_p + w_p}{l}\right) \\
&= \left(\frac{1 + \sqrt{2}}{l}\right)\left(\frac{\pi}{l}\right).
\end{aligned}
$$

For $d = -pl$, we have $n = 1$, $m = 0$. Thus

$$
\begin{aligned}
(-d, u+w)_l &= \left(\frac{u_{-1} + w_{-1}}{l}\right)^0 \left(\frac{u_2 + w_2}{l}\right)^0 \left(\frac{u_p + w_p}{l}\right) \\
&= \left(\frac{\pi}{l}\right).
\end{aligned}
$$

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