MA2071

Solutions to Practice midterm

1) (i) We have $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ or $ax+by+cz+d=0$ where $d=-(ax_0+by_0+cz_0)$.

(ii) Let $P_0=(x_0,y_0,z_0)\in P$ and consider the vector $\mathbf{b}=\overrightarrow{P_0P_1}=\langle x_1-x_0,y_1-y_0,z_1-z_0\rangle$. Note that D equals the absolute value of the scalar projection of $\mathbf{b}$ onto the normal vector $\mathbf{n}=\langle a,b,c\rangle$. Thus,

$$D=|\mathrm{comp}_{\mathbf{n}}\mathbf{b}|=\frac{|\mathbf{n}\cdot\mathbf{b}|}{|\mathbf{n}|}=\frac{|(ax_1+by_1+cz_1)-(ax_0+by_0+cz_0)|}{\sqrt{a^2+b^2+c^2}}.$$

As $P_0\in P$, we have $ax_0+by_0+cz_0+d=0$ and the result follows.

2) (i) Let (x,y) approach $(0,0)$ through the line L given by $y=mx$ where $m\neq 0$. Check that $f(x,y)=f(x,mx)=\dfrac{m^2x}{1+m^4x^2}$. So, as (x,y) approaches $(0,0)$ through L, then $f(x,y)$ approaches 0. Now, let (x,y) approach $(0,0)$ along the parabola P given by $x=y^2$. Check that $f(x,y)=f(y^2,y)=\frac{1}{2}$. So, as (x,y) approaches $(0,0)$ through P, $f(x,y)$ approaches $\frac{1}{2}$. Thus, the limit does not exist.

(ii) Let $F(x,y,z)=xyz-\cos(x+y+z)=0$. So, $\dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z}=-\dfrac{yz+\sin(x+y+z)}{xy+\sin(x+y+z)}$ and $\dfrac{\partial z}{\partial y}=-\dfrac{F_y}{F_z}=-\dfrac{xz+\sin(x+y+z)}{xy+\sin(x+y+z)}$.

3) (i) See the notes.

(ii) For $(x,y)\neq(0,0)$, check that $f_x(x,y)=\dfrac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}$ and $f_y(x,y)=\dfrac{x^5-4x^3q^2-xy^4}{(x^2+y^2)^2}$. Using the definition, check that $f_x(0,0)=0$ and $f_y(0,0)=0$. Also, use the definition to check that $f_{xy}(0,0)=-1$ and $f_{yx}(0,0)=1$. Note that $f_{xy}(x,y)$ is not continuous at $(0,0)$. To see this, check that $f_{xy}=\dfrac{x^6+9x^4y^2-4x^2y^4+4y^6}{(x^2+y^2)^3}$. Now, as (x,y) approaches $(0,0)$ along the x-axis, then $f_{xy}(x,y)$ approaches 1 while as (x,y) approaches $(0,0)$ along the y-axis, then $f_{xy}(x,y)$ approaches 4. Thus, Clairaut's Theorem does not apply.

(iii) Let $u=x+at$ and $v=x-at$. Then $z=f(u)+g(v)$. So, $\dfrac{\partial z}{\partial u}=f'(u)$ and $\dfrac{\partial z}{\partial v}=g'(v)$. Thus, $\dfrac{\partial z}{\partial t}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial t}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial t}=af'(u)-ag'(v)$ and

$$\frac{\partial^2 z}{\partial t^2} = a\frac{\partial}{\partial t}(f'(u) - g'(v)) = a\left(\frac{df'(u)}{du}\frac{\partial u}{\partial t} - \frac{dg'(v)}{dv}\frac{\partial u}{\partial t}\right) = a^2 f''(u) + a^2 g''(v).$$

Similarly, $\dfrac{\partial z}{\partial x} = f'(u) + g'(v)$ and $\dfrac{\partial^2 z}{\partial x^2} = f''(u) + g''(v)$. So, the result follows.

4) (i) The direction of fastest change is $\nabla f(x, y) = (2x - 2)\mathbf{i} + (2y - 4)\mathbf{j}$. So, we need to find all points (x, y) where $\nabla f(x, y)$ is parallel to $\mathbf{i} + \mathbf{j}$. This occurs if and only if $(2x - 2)\mathbf{i} + (2y - 4)\mathbf{j} = k(\mathbf{i} + \mathbf{j})$ for some $k \in \mathbb{R}$. Thus, $k = 2x - 2$ and $k = 2y - 4$ and so $y = x + 1$. So, the direction of fastest change is $\mathbf{i} + \mathbf{j}$ at all points on the line $y = x + 1$.

(ii) Let x, y and z be dimensions of the rectangular box. Then $V = xyz$ and $L = \sqrt{x^2 + y^2 + z^2}$. So, $L^2 = x^2 + y^2 + z^2$ and so $z = \sqrt{L^2 - x^2 - y^2}$. This gives $V(x, y) = xy\sqrt{L^2 - x^2 - y^2}$ for x, $y > 0$. Check that
$V_x = y\sqrt{L^2 - x^2 - y^2} - \dfrac{x^2 y}{\sqrt{L^2 - x^2 - y^2}}$ and $V_y = x\sqrt{L^2 - x^2 - y^2} - \dfrac{xy^2}{\sqrt{L^2 - x^2 - y^2}}$.
Now, $V_x = 0$ implies $y(L^2 - x^2 - y^2) = x^2 y$ or $y(L^2 - 2x^2 - y^2) = 0$ and so $2x^2 + y^2 = L^2$ since $y > 0$. Similarly, $V_y = 0$ implies $x(L^2 - x^2 - y^2) = xy^2$ or $x(L^2 - x^2 - 2y^2) = 0$ and so $x^2 + 2y^2 + L^2$ since $x > 0$. Substitute $y^2 = L^2 - 2x^2$ into $x^2 + 2y^2 = L^2$ to get $x^2 + 2L^2 - 4x^2 = L^2$ or $3x^2 = L^2$. Thus, $x = \frac{L}{\sqrt{3}}$ since $x > 0$. This implies $y = \frac{L}{\sqrt{3}}$. So, the only critical point is $\left(\frac{L}{\sqrt{3}}, \frac{L}{\sqrt{3}}\right)$ which must be the absolute maximum. Thus, the largest possible volume is $V\left(\dfrac{L}{\sqrt{3}}, \dfrac{L}{\sqrt{3}}\right) = \dfrac{L^3}{3\sqrt{3}}$.