MA2071

Solutions to Midterm Exam

1) (i) We have $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ or $ax+by+cz+d=0$ where $d=-(ax_0+by_0+cz_0)$.

(ii) True.

(iii) First, note that the planes are parallel as their normal vectors $\langle 10,2,-2\rangle$ and $\langle 5,1,-1\rangle$ are parallel. To find D, choose any point on one plane and calculate the distance from it to the other plane. Putting $y=z=0$ into the first equation yields $10x=5$ and so the point $(1/2,0,0)$ is on this plane. By the lecture notes (or practice midterm), the distance between $(1/2,0,0)$ and the plane $5x+y-z-1=0$ is

$$D=\frac{|5(1/2)+1(0)-1(0)-1|}{\sqrt{5^2+1^2+(-1)^2}}=\frac{\sqrt{3}}{6}.$$

2) (i) See the lecture notes.

(ii) Let $u=x-y$. Then $\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x}=\dfrac{\partial z}{\partial u}\cdot 1$. Also, we have $\dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial y}=\dfrac{\partial z}{\partial u}\cdot(-1)$. Thus, the result follows.

3) (i) We have $f_x=2x+y\cos xy$ and so $f_x(1,0)=2$. Also, $f_y=x\cos xy$ and so $f_y(1,0)=1$. Now, if $\mathbf{u}$ is a unit vector with angle θ with the positive x-axis, then $D_{\mathbf{u}}f(1,0)=f_x(1,0)\cos\theta+f_y(1,0)\sin\theta=2\cos\theta+\sin\theta$. We want $1=D_{\mathbf{u}}f(1,0)$ which implies $1=2\cos\theta+\sin\theta$. Squaring both sides yields $\sin^2\theta=(1-2\cos\theta)^2$ or $1-\cos^2\theta=1-4\cos\theta+4\cos^2\theta$ or $\cos\theta(5\cos\theta-4)=0$. This implies $\cos\theta=0$ or $\cos\theta=4/5$. Thus, $\theta=\pi/2$ or $\theta=2\pi-\cos^{-1}(4/5)=5.64$.

(ii) Let $F(x,y,z)=yz-\ln(x+z)$. Then $\nabla F(x,y,z)=\left\langle -\dfrac{1}{x+z}, z, y-\dfrac{1}{x+z}\right\rangle$. So, $\nabla F(0,0,1)=\langle -1,1,-1\rangle$. Thus, we have $-1(x-0)+(1)(y-0)-1(z-1)=0$ or $x-y+z=1$.

4) Let x and y denote the sides of the rectangle. Then $f(x,y)=xy$ and $g(x,y)=2x+2y=p$. Using Lagrange multipliers, we have $f_x=\lambda g_x$ or $y=2\lambda$ and $f_y=\lambda g_y$ or $x=2\lambda$. Thus, $\lambda=y/2=x/2$ or $x=y$. Thus, we have a square. Note that the length of a side is then $4x=p$ or $x=p/4$.