Exercise Set 9

(1) Find a potential function for the vector field $F(x,\, y) = x^3y^4\,\mathbb{i} + x^4y^3\,\mathbb{j}$ and use it to evaluate the line integral of F along the curve $C(t) = \sqrt{t}\,\mathbb{i} + (1+t^3)\,\mathbb{j}$ for $t \in [0,\, 1]$.

(2) Find a potential function for the vector field $F(x,\, y,\, z) = yz\,\mathbb{i} + xz\,\mathbb{j} + (xy + 2z)\,\mathbb{k}$ and use it to evaluate the line integral of F along the line segment from $(1,\, 0,\, -2)$ to $(4,\, 6,\, 3)$.

(3) Using Green's theorem, evaluate

$$\int_C (3y - e^{\sin x})\, dx + (7x + \sqrt{y^4+1})\, dy$$

where C is the circle $x^2 + y^2 = 1$.

(4) Using Green's theorem, evaluate

$$\int_C (y + e^{\sqrt{x}})\, dx + (2x + \cos y^2)\, dy$$

where C is the positively oriented boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

(5) Find (a) the curl, and (b) the divergence of the following two vector fields:

(i) $F(x,\, y,\, z) = xyz\,\mathbb{i} - x^2y\,\mathbb{k}$
(ii) $F(x,\, y,\, z) = e^x \sin y\,\mathbb{i} + e^x \cos y\,\mathbb{j} + z\,\mathbb{k}$

(6) Prove that for a function $f : \mathbb{R}^3 \to \mathbb{R}$ we have $\operatorname{curl}(\nabla f) = 0$.

(7) Show that for a vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$ we have $\operatorname{div}(\operatorname{curl}(F)) = 0$.

(8) Determine whether the vector field $F(x,\, y,\, z) = 2xy\,\mathbb{i} + (x^2 + 2yz)\,\mathbb{j} + y^2\,\mathbb{k}$ is conservative and, if so, find a potential function.