1 i) $f(x,y) = x^4 + y^4 - 4xy + 2$

$f_x(x,y) = 4x^3 - 4y$

$f_y(x,y) = 4y^3 - 4x$

$f_{xx}(x,y) = 12x^2$

$f_{xy}(x,y) = -4$

$f_{yy}(x,y) = 12y^2$

$f_x(x,y) = 0 \implies x^3 = y$

$f_y(x,y) = 0 \implies y^3 = x$

If both are 0, then $x^9 = x$

$\implies x = 1$ or $x = 0$ or $x = -1$

$\hookrightarrow y = 1$ $\quad \hookrightarrow y = 0$ $\quad \hookrightarrow y = -1$

$\implies$ The critical points are $(1,1)$, $(0,0)$ and $(-1,-1)$.

$$D = \det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \det\begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix} = 144x^2y^2 - 16$$

At $(0,0)$: $D = -16 < 0 \implies$ saddle point

At $(1,1)$: $D = 128 > 0$

$f_{xx} = 12 > 0 \implies$ local minimum

At $(-1,-1)$: $D = 128 > 0$

$f_{xx} = 12 > 0 \implies$ local minimum.

ii) $f(x,y) = x \sin y$

$f_x(x,y) = \sin y$

$f_y(x,y) = x \cos y$

$f_{xx}(x,y) = 0$

$f_{xy}(x,y) = \cos y$

$f_{yy}(x,y) = -x \sin y$

$f_x = 0 \implies y = n\pi$ for some $n \in \mathbb{Z}$

$\implies \cos y \neq 0$

$f_y = 0 \implies x = 0$ or $\cos y = 0$

doesn't happen

$\implies$ The critical points are $(0, n\pi)$ for $n \in \mathbb{Z}$.

$$D = \det\begin{pmatrix} 0 & \cos y \\ \cos y & -x\sin y \end{pmatrix} = -\cos^2 y < 0.$$

$\Longrightarrow$ $(0, n\pi)$ is a saddle point for all $n \in \mathbb{Z}$.

iii) $f(x,y) = x^2 y e^{-x^2-y^2}$

$f_x(x,y) = 2xy e^{-x^2-y^2} - 2x^3 y e^{-x^2-y^2}$

$= 2xy(1-x^2)e^{-x^2-y^2}$

$f_y(x,y) = x^2 e^{-x^2-y^2} - 2y^2x^2 e^{-x^2-y^2}$

$= x^2(1-2y^2)e^{-x^2-y^2}$

$f_{xx}(x,y) = 2y(1-x^2)e^{-x^2-y^2} - 4x^2 y e^{-x^2-y^2} - 4x^2 y(1-x^2)e^{-x^2-y^2}$

$= 2y(1-5x^2+2x^4)e^{-x^2-y^2}$

$f_{xy}(x,y) = 2x(1-x^2)e^{-x^2-y^2} - 4xy^2(1-x^2)e^{-x^2-y^2}$

$= 2x(1-x^2)(1-2y^2)e^{-x^2-y^2}$

$f_{yy}(y,y) = -4x^2 y e^{-x^2-y^2} - 2x^2 y e^{-x^2-y^2}(1-2y^2)$

$= -2x^2 y(3-2y^2)e^{-x^2-y^2}$

$f_x = 0 \Rightarrow x = 0$ or $y = 0$ or $1 - x^2 = 0$

$\Rightarrow x = 1$ or $x = -1$

$f_y = 0 \Rightarrow x = 0$ or $1 - 2y^2 = 0$

$\Rightarrow y = \pm \frac{1}{\sqrt{2}}$

$\rightarrow$ The critical points are $(0,y)$ for all y,

$(1, \frac{1}{\sqrt{2}})$, $(1, -\frac{1}{\sqrt{2}})$, $(-1, \frac{1}{\sqrt{2}})$, $(-1, -\frac{1}{\sqrt{2}})$

At $(0,y)$: $D = \det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = 0$

$\Longrightarrow$ Second derivatives test tells us nothing.

$f(x,y) > 0$	$f(x,y) > 0$
$f(x,y) < 0$	$f(x,y) < 0$

$(0,y)$ is min if $y > 0$

saddle if $y = 0$

max if $y < 0$.

On the axes: $f(x,y) = 0$

At $(1, \frac{1}{\sqrt{2}})$: $D = 8e^{-3} > 0$

$f_{xx} = -2\sqrt{2}\, e^{-\frac{3}{2}} < 0$

$\Longrightarrow (1, \frac{1}{\sqrt{2}})$ is a local maximum.

At $(1, -\frac{1}{\sqrt{2}})$: $D = 8e^{-3} > 0$

$f_{xx} = 2\sqrt{2}\, e^{-\frac{3}{2}} > 0$

$\Longrightarrow (1, -\frac{1}{\sqrt{2}})$ is a local minimum.

At $(-1, \frac{1}{\sqrt{2}})$: $D = 8e^{-3} > 0$

$f_{xx} = -2\sqrt{2}\, e^{-\frac{3}{2}} < 0$

$\Longrightarrow (-1, \frac{1}{\sqrt{2}})$ is a local maximum.

At $(-1, -\frac{1}{\sqrt{2}})$: $D = 8e^{-3} > 0$

$f_{xx} = 2\sqrt{2}\, e^{-\frac{3}{2}} > 0$

$\Longrightarrow (-1, -\frac{1}{\sqrt{2}})$ is a local minimum.

2) $z^2 = xy + 1$, find point closest to origin.

I.e. minimise $x^2 + y^2 + z^2 = x^2 + y^2 + xy + 1$.

$$\left.\begin{aligned} f_x(x,y) &= 2x + y = 0 \Longleftrightarrow y = -2x \\ f_y(x,y) &= 2y + x = 0 \Longleftrightarrow x = -2y \end{aligned}\right\} x = 4x$$

$\Longrightarrow x = 0$ and $y = 0$.

$x = y = 0 \Longrightarrow z^2 = 1 \Longrightarrow z = \pm 1$

$\Longrightarrow$ The points closest to the origin are $(0,0,1)$ and $(0,0,-1)$.

3) Find $x, y, z > 0$ s.t. $x + y + z = 100$ and xyz is maximal.

$x + y + z = 100 \longrightarrow z = 100 - x - y$

$\Longrightarrow$ Want to maximise $f(x,y) = xy(100 - x - y)$

$f_x(x,y) = 100y - 2xy - y^2$

$f_y(x,y) = 100x - x^2 - 2xy$

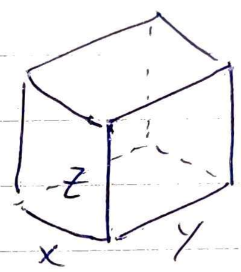
$f_x = 0 \implies y = 0$ or $100 - 2x - y = 0$
$f_y = 0 \implies x = 0$ or $100 - x - 2y = 0$

$$\Leftrightarrow \begin{cases} 2x + y = 100 \\ x + 2y = 100 \end{cases} \implies x - y = 0$$
$$\implies x = y$$
$$\implies 3x = 100$$

$$\implies x = y = \frac{100}{3}$$
$$\implies z = \frac{100}{3}$$

$\implies xyz$ is maximal if $x = y = z = \frac{100}{3}$
(and then $xyz = \frac{1000000}{27}$)

4) Find $x, y, z > 0$ s.t.
$4(x + y + z) = c$ and
xyz is maximal.



Similarly to problem 3, xyz is maximal
if $x = y = z = \frac{c}{12}$.
(and then $xyz = \frac{c^3}{1728}$).

5) $xyz = 32\,000$
Minimise $xy + 2xz + 2yz$

$$z = \frac{32\,000}{xy}$$
$\longrightarrow$ Want to minimise $f(x,y) = xy + \frac{64000}{y} + \frac{64000}{x}$

$f_x(x,y) = y - \frac{64000}{x^2} = 0$ if $y = \frac{64000}{x^2}$
$f_y(x,y) = x - \frac{64000}{y^2} = 0$ if $x = \frac{64000}{y^2}$

$\implies x^2 y = xy^2 = 64\,000$
Divide by $xy \implies x = y$
$\implies x = \frac{64000}{x^2} \implies x^3 = 64\,000$
$\implies x = 40$ and $y = 40$.

$$z = \frac{32000}{xy} = \frac{32000}{40^2} = 20$$

$$\Rightarrow (x, y, z) = (40, 40, 20).$$

6) $L = \sqrt{x^2 + y^2 + z^2}$, maximise xyz.
Using Lagrange multipliers: let
$f(x,y,z) = xyz$, $g(x,y,z) = x^2 + y^2 + z^2$
Maximise $f(x,y,z)$ subject to $g(x,y,z) = L^2$.

At the maximum, $\nabla f = \lambda \nabla g$ for some λ.
$\nabla f = (yz, xz, xy)$
$\nabla g = (2x, 2y, 2z)$

$$\Rightarrow \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} = \begin{pmatrix} 2\lambda x \\ 2\lambda y \\ 2\lambda z \end{pmatrix}$$

$$\lambda = \frac{yz}{2x} = \frac{xz}{2y} = \frac{xy}{2z}$$

$$\Rightarrow \frac{xyz}{2\lambda} = x^2 = y^2 = z^2$$

$$\Rightarrow x = y = z = \frac{L}{\sqrt{3}}$$

$$\Rightarrow xyz = \frac{L^3}{3\sqrt{3}}.$$