1i) $\|\underline{v}\| = \sqrt{4^2+(-3)^2} = 5 \longrightarrow \frac{\underline{v}}{\|\underline{v}\|} = \left(\frac{4}{5}, \frac{-3}{5}\right)$

$\nabla f = \left(2\sqrt{y}, \frac{x}{\sqrt{y}}\right)$

$D_{\underline{v}} f(x,y) = \nabla f \cdot \frac{\underline{v}}{\|\underline{v}\|} = \frac{8}{5}\sqrt{y} - \frac{3}{5}\frac{x}{\sqrt{y}}$

At $P=(3,4)$: $\frac{8}{5}\sqrt{4} - \frac{3}{5}\cdot\frac{3}{\sqrt{4}} = \frac{23}{10}$.

ii) $\|\underline{v}\| = \sqrt{1^2+2^2+3^2} = \sqrt{14} \longrightarrow \frac{\underline{v}}{\|\underline{v}\|} = \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$

$\nabla f = \left(\frac{1}{y+z}, -\frac{x}{(y+z)^2}, -\frac{x}{(y+z)^2}\right)$

$D_{\underline{v}} f(x,y) = \frac{1}{\sqrt{14}(y+z)} - \frac{2x}{\sqrt{14}(y+z)^2} - \frac{3x}{\sqrt{14}(y+z)^2} = \frac{y+z-5x}{\sqrt{14}(y+z)^2}$

At $P=(4,1,1)$: $\frac{1+1-5\cdot4}{\sqrt{14}(1+1)^2} = -\frac{9}{2\sqrt{14}}$

2) $\underline{v} = (-2,-1,-3)$ $\quad \|\underline{v}\| = \sqrt{14}$ $\quad \frac{\underline{v}}{\|\underline{v}\|} = \left(\frac{-2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{-3}{\sqrt{14}}\right)$

$\nabla f = (2x, 2y, 2z)$

$D_{\underline{v}} f = -\frac{4x}{\sqrt{14}} - \frac{2y}{\sqrt{14}} - \frac{6z}{\sqrt{14}}$

At $P=(2,1,3)$: $\frac{-8-2-18}{\sqrt{14}} = \frac{-28}{\sqrt{14}} = -2\sqrt{14}$

3i) $\nabla f = (2xy^3z^4, 3x^2y^2z^4, 4x^2y^3z^3)$

At $P=(1,1,1)$: $\nabla f(P) = (2,3,4)$.

Maximal rate of change occurs in direction of $\nabla f(P)$ and equals $\|\nabla f(P)\| = \sqrt{2^2+3^2+4^2} = \sqrt{29}$.

Direction : $\frac{\nabla f(P)}{\|\nabla f(P)\|} = \left(\frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}\right)$.

The minimal rate of change occurs in the opposite direction, i.e. $\left(-\frac{2}{\sqrt{29}}, -\frac{3}{\sqrt{29}}, -\frac{4}{\sqrt{29}}\right)$, and equals $-\sqrt{29}$.

3 ii) $\cancel{\nabla f = \left(\frac{1}{xy^2z^3}, \frac{2y}{xy^2z^3}, \frac{3z^2}{xy^2z^3}\right)}$

$f(x,y,z) = \ln(xy^2z^3) = \ln(x) + 2\ln(y) + 3\ln(z)$

$\nabla f = \left(\frac{1}{x}, \frac{2}{y}, \frac{3}{z}\right)$

At $P = (1, -2, -3)$: $\nabla f(P) = (1, -1, -1)$

$\|\nabla f(P)\| = \sqrt{3}$

Maximal rate of change: $\sqrt{3}$ in direction $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$

Minimal rate of change: $-\sqrt{3}$ in direction $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

4 i) $\nabla f = (2x, 4y, 6z)$

$\nabla f(P) = \cancel{(\ldots)}\ (8, -4, 6)$

$\nabla f(P)$ is perpendicular to the tangent surface

$\Longrightarrow$ The tangent plane is given by

$\nabla f(P) \cdot (x - x_0, y - y_0, z - z_0) = 0$

$(8, -4, 6) \cdot (x - 4, y + 1, z - 1) = 0$

$8x - 32 - 4y - 4 + 6z - 6 = 0$

$8x - 4y + 6z = 42$

ii) $f(x,y,z) = x - z - 4\arctan(yz) = 0$

$\nabla f(x,y,z) = \left(1, \frac{-4z}{1+(yz)^2}, -1 - \frac{4y}{1+(yz)^2}\right)$

$\nabla f(1+\pi, 1, 1) = (1, -2, -3)$

The tangent plane is given by

$(1, -2, -3) \cdot (x - (1+\pi), y - 1, z - 1) = 0$

$x - 1 - \pi - 2y + 2 - 3z + 3 = 0$

$x - 2y - 3z = \pi - 4$

5) Planes are parallel $\Longleftrightarrow$ normal vectors are parallel.

$$\nabla(x^2+2y^2+3z^2) = (2x, 4y, 6z)$$

is a normal vector to the ellipsoid.

$\underline{n} = (3, -1, 3)$ is a normal vector to the plane $3x - y + 3z = 1$.

$\longrightarrow$ We're looking for points (x, y, z) where

$(2x, 4y, 6z) = c(3, -1, 3)$ for some c

and $x^2+2y^2+3z^2 = 1$.

$$2x = 3c \Longrightarrow x = \frac{3c}{2}$$

$$4y = -c \Longrightarrow y = -\frac{c}{4}$$

$$6z = 3c \Longrightarrow z = \frac{c}{2}$$

Substituting into equation gives

$$\left(\frac{3c}{2}\right)^2 + 2\left(-\frac{c}{4}\right)^2 + 3\left(\frac{c}{2}\right)^2 = 1$$

$$\frac{9}{4}c^2 + \frac{1}{8}c^2 + \frac{3}{4}c^2 = 1$$

$$\frac{25}{8}c^2 = 1 \Longrightarrow c = \pm\sqrt{\frac{8}{25}} = \pm\frac{2}{5}\sqrt{2}$$

$c = \frac{2}{5}\sqrt{2}$ corresponds to the point $\left(\frac{3}{5}\sqrt{2}, -\frac{1}{10}\sqrt{2}, \frac{1}{5}\sqrt{2}\right)$

$c = -\frac{2}{5}\sqrt{2}$ corresponds to $\left(-\frac{3}{5}\sqrt{2}, \frac{1}{10}\sqrt{2}, -\frac{1}{5}\sqrt{2}\right)$