1 i) Recall the equation for the tangent plane at (x_0, y_0, z_0):

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

$$\frac{\partial z}{\partial x} = \frac{1}{2\sqrt{4 - x^2 - 2y^2}} \cdot -2x = -\frac{x}{z}$$

at $p = (1, -1, 1)$: $\frac{\partial z}{\partial x} = -1$

$$\frac{\partial z}{\partial y} = \frac{1}{2\sqrt{4 - x^2 - 2y^2}} \cdot -4y = -\frac{2y}{z}$$

at $p = (1, -1, 1)$: $\frac{\partial z}{\partial y} = 2$

$\Rightarrow$ Tangent plane: $z - 1 = -(x-1) + 2(y+1)$

$$z = -x + 2y + 4$$

ii) $\frac{\partial z}{\partial x} = \frac{y}{x}$, at $p = (1, 4, 0)$: $\frac{\partial z}{\partial x} = 4$

$\frac{\partial z}{\partial y} = \log(x)$, at $p = (1, 4, 0)$: $\frac{\partial z}{\partial y} = 0$.

$\Rightarrow$ Tangent plane: $z = 4(x-1) = 4x - 4$.

2 i) $$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$= 3x^2 \log(y^2)\, dx + \frac{2x^3 y}{y^2} dy$$

$$= 3x^2 \log(y^2)\, dx + \frac{2x^3}{y} dy$$

ii) $$\frac{\partial w}{\partial x} = \frac{1}{\sqrt{x^2+y^2+z^2}} \cdot \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2x$$

$$= \frac{x}{x^2+y^2+z^2}$$

Similarly, $\frac{\partial w}{\partial y} = \frac{y}{x^2+y^2+z^2}$ and $\frac{\partial w}{\partial z} = \frac{z}{x^2+y^2+z^2}$

$$\Rightarrow dw = \frac{x}{x^2+y^2+z^2} dx + \frac{y}{x^2+y^2+z^2} dy + \frac{z}{x^2+y^2+z^2} dz.$$

3 i) $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

$\frac{\partial z}{\partial x} = \cos x \cos y = \cos(\pi t)\cos\sqrt{t}$

$\frac{\partial z}{\partial y} = -\sin x \sin y = -\sin(\pi t)\sin\sqrt{t}$

$\frac{dx}{dt} = \pi$

$\frac{dy}{dt} = \frac{1}{2\sqrt{t}}$

$\Longrightarrow \frac{dz}{dt} = \pi\cos(\pi t)\cos\sqrt{t} - \frac{\sin(\pi t)\sin\sqrt{t}}{2\sqrt{t}}$

ii) $\frac{\partial w}{\partial x} = y = e^t \sin t$

$\frac{\partial w}{\partial y} = x + z^2 = e^t + e^{2t}\cos^2 t$

$\frac{\partial w}{\partial z} = 2yz = 2e^{2t}\cos t \sin t$

$\frac{dx}{dt} = e^t$

$\frac{dy}{dt} = e^t \sin t + e^t \cos t$

$\frac{dz}{dt} = e^t \cos t - e^t \sin t$

$\Longrightarrow \frac{dw}{dt} = e^{2t}\sin t + (e^t + e^{2t}\cos^2 t)e^t(\sin t + \cos t)$
$+ 2e^{2t}\cos t \sin t\, e^t(\cos t - \sin t)$
$= e^{2t}\sin t + e^{2t}\sin t + e^{2t}\cos t + e^{3t}\cos^2 t \sin t$
$+ e^{3t}\cos^3 t + 2e^{3t}\cos^2 t \sin t - 2e^{3t}\cos t \sin^2 t$
$= e^{2t}(2\sin t + \cos t) + e^{3t}(3\cos^2 t \sin t + \cos^3 t$
$- 2\cos t \sin^2 t)$

4 i) $\frac{\partial z}{\partial x} = 2x + y^3 = 31$ $\qquad$ $\frac{\partial z}{\partial y} = 3xy^2 = 54$

$\frac{\partial x}{\partial u} = v^2 = 1$ $\qquad$ $\frac{\partial y}{\partial u} = 1$

$\frac{\partial x}{\partial v} = 2uv = 4$ $\qquad$ $\frac{\partial y}{\partial v} = e^w = 1$

$\frac{\partial x}{\partial w} = 3w^2 = 0$ $\qquad$ $\frac{\partial y}{\partial w} = ve^w = 1$

$x = 2$ $\qquad$ $y = 3$

$\frac{\partial z}{\partial u} = (2x + y^3)v^2 + 3xy^2 = 85$

$\frac{\partial z}{\partial v} = (2x + y^3)2uv + 3xy^2 e^w = 178$

$\frac{\partial z}{\partial w} = (2x + y^3)\cdot 0 + 3xy^2 v e^w = 54$

4ii) $\frac{\partial R}{\partial u} = \frac{2u}{u^2+v^2+w^2} = \frac{6}{14} = \frac{3}{7}$

$\frac{\partial R}{\partial v} = \frac{2v}{u^2+v^2+w^2} = \frac{2}{14} = \frac{1}{7}$

$\frac{\partial R}{\partial w} = \frac{2w}{u^2+v^2+w^2} = \frac{4}{14} = \frac{2}{7}$

$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial x} = 2 \qquad \frac{\partial w}{\partial x} = 2y = 2$

$\frac{\partial u}{\partial y} = 2 \qquad \frac{\partial v}{\partial y} = -1 \qquad \frac{\partial w}{\partial y} = 2x = 2$

$u = x+2y = 3 \qquad v = 2x-y = 1 \qquad w = 2xy = 2$

$\frac{\partial R}{\partial x} = \frac{\partial R}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial R}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial R}{\partial w}\frac{\partial w}{\partial x} = \frac{3}{7} + \frac{2}{7} + \frac{4}{7} = \frac{9}{7}$

$\frac{\partial R}{\partial y} = \frac{\partial R}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial R}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial R}{\partial w}\frac{\partial w}{\partial y} = \frac{6}{7} - \frac{1}{7} + \frac{4}{7} = \frac{9}{7}$

5ia) Note that $x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$

and $\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta \implies \theta = \arctan\frac{y}{x}$.

$\implies \frac{\partial}{\partial x} r^2 = \frac{\partial}{\partial x}(x^2+y^2)$

$2r\frac{\partial r}{\partial x} = 2x \implies \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$

$\frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot -\frac{y}{x^2} = \frac{-y}{x^2+y^2} = -\frac{y}{r^2} = -\frac{\sin\theta}{r}$

$\frac{\partial \theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2} = \frac{x}{r^2} = \frac{\cos\theta}{r}$

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial \theta}{\partial \theta}$

$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta\frac{\partial z}{\partial r} + \frac{\cos\theta}{r}\frac{\partial z}{\partial \theta}$

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \cos^2\theta\left(\frac{\partial z}{\partial r}\right)^2 - 2\frac{\sin\theta\cos\theta}{r}\frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta} + \frac{\sin^2\theta}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$
$$+ \sin^2\theta\left(\frac{\partial z}{\partial r}\right)^2 + 2\frac{\sin\theta\cos\theta}{r}\frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta} + \frac{\cos^2\theta}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

$$= \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

5:b) $$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\cos\theta \frac{\partial z}{\partial r} - \frac{y}{r^2}\frac{\partial z}{\partial \theta}\right)$$

$$= -\sin\theta \frac{\partial \theta}{\partial x}\frac{\partial z}{\partial r} + \cos\theta\left(\frac{\partial^2 z}{\partial r^2}\frac{\partial r}{\partial x} + \frac{\partial^2 z}{\partial r \partial \theta}\frac{\partial \theta}{\partial x}\right) + 2\frac{y}{r^3}\frac{\partial r}{\partial x}\frac{\partial z}{\partial \theta} - \frac{y}{r^2}\left(\frac{\partial^2 z}{\partial \theta \partial r}\frac{\partial r}{\partial x} + \frac{\partial^2 z}{\partial \theta^2}\frac{\partial \theta}{\partial x}\right)$$

$$= \frac{\sin^2\theta}{r}\frac{\partial z}{\partial r} + \cos^2\theta \frac{\partial^2 z}{\partial r^2} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 z}{\partial r \partial \theta} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial z}{\partial \theta} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 z}{\partial \theta \partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 z}{\partial \theta^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\sin\theta \frac{\partial z}{\partial r} + \frac{x}{r^2}\frac{\partial z}{\partial \theta}\right)$$

$$= \cos\theta \frac{\partial \theta}{\partial y}\frac{\partial z}{\partial r} + \sin\theta\left(\frac{\partial^2 z}{\partial r^2}\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial r \partial \theta}\frac{\partial \theta}{\partial y}\right) - 2\frac{x}{r^3}\frac{\partial r}{\partial y}\frac{\partial z}{\partial \theta} + \frac{x}{r^2}\left(\frac{\partial^2 z}{\partial \theta \partial r}\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial \theta^2}\frac{\partial \theta}{\partial y}\right)$$

$$= \frac{\cos^2\theta}{r}\frac{\partial z}{\partial r} + \sin^2\theta \frac{\partial^2 z}{\partial r^2} + \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 z}{\partial r \partial \theta} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial z}{\partial \theta} + \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 z}{\partial \theta \partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 z}{\partial \theta^2}$$

$$\Longrightarrow \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{r}\frac{\partial z}{\partial r} + \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}$$

5 ii) $\cancel{s = \frac{x+y}{2}}$ $\cancel{t = \frac{x-y}{2}}$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}$$

$$\longrightarrow \frac{\partial z}{\partial s}\frac{\partial z}{\partial t} = \left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right)\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2$$