

$$\begin{aligned}
 \text{i)} \quad f(x, y) &= x^4 + y^4 - 4xy + 2 \\
 f_x(x, y) &= 4x^3 - 4y \\
 f_y(x, y) &= 4y^3 - 4x \\
 f_{xx}(x, y) &= 12x^2 \\
 f_{xy}(x, y) &= -4 \\
 f_{yy}(x, y) &= 12y^2
 \end{aligned}$$

$$f_x(x, y) = 0 \implies x^3 = y$$

$$f_y(x, y) = 0 \implies y^3 = x$$

If both are 0, then $x^3 = x$

$$\implies x = 1 \quad \text{or} \quad x = 0 \quad \text{or} \quad x = -1$$

$$\hookrightarrow y = 1$$

$$\hookrightarrow y = 0$$

$$\hookrightarrow y = -1$$

$\implies$ The critical points are $(1, 1)$, $(0, 0)$ and $(-1, -1)$.

$$D = \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \det \begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix} = 144x^2y^2 - 16$$

$$\text{At } (0, 0): D = -16 < 0 \implies \text{saddle point}$$

$$\text{At } (1, 1): D = 128 > 0$$

$$f_{xx} = 12 > 0 \implies \text{local minimum}$$

$$\text{At } (-1, -1): D = 128 > 0$$

$$f_{xx} = 12 > 0 \implies \text{local minimum}$$

$$\text{ii)} \quad f(x, y) = x \sin y$$

$$f_x(x, y) = \sin y$$

$$f_y(x, y) = x \cos y$$

$$f_{xx}(x, y) = 0$$

$$f_{xy}(x, y) = \cos y$$

$$f_{yy}(x, y) = -x \sin y$$

$$f_x = 0 \implies y = n\pi$$

for some $n \in \mathbb{Z}$

$$\implies \cos y \neq 0$$

$$f_y = 0 \implies x = 0 \quad \text{or} \quad \cos y = 0$$

$\downarrow$
doesn't happen

$\implies$ The critical points are $(0, n\pi)$ for $n \in \mathbb{Z}$.

$$D = \det \begin{pmatrix} 0 & \cos y \\ \cos y & -x \sin y \end{pmatrix} = \cancel{\cos^2 y} - \cos^2 y < 0.$$

$\Rightarrow (0, n\pi)$ is a saddle point for all $n \in \mathbb{Z}$.

$$\begin{aligned} \text{iii)} \quad f(x, y) &= x^2 y e^{-x^2 - y^2} \\ f_x(x, y) &= 2xy e^{-x^2 - y^2} - 2x^3 y e^{-x^2 - y^2} \\ &= 2xy(1 - x^2) e^{-x^2 - y^2} \\ f_y(x, y) &= x^2 e^{-x^2 - y^2} - 2y^2 x^2 e^{-x^2 - y^2} \\ &= x^2(1 - 2y^2) e^{-x^2 - y^2} \\ f_{xx}(x, y) &= 2y(1 - x^2) e^{-x^2 - y^2} - 4x^2 y e^{-x^2 - y^2} - 4x^2 y(1 - x^2) e^{-x^2 - y^2} \\ &= 2y(1 - 5x^2 + 2x^4) e^{-x^2 - y^2} \\ f_{xy}(x, y) &= 2x(1 - x^2) e^{-x^2 - y^2} - 4xy^2(1 - x^2) e^{-x^2 - y^2} \\ &= 2x(1 - x^2)(1 - 2y^2) e^{-x^2 - y^2} \\ f_{yy}(x, y) &= -4x^2 y e^{-x^2 - y^2} - 2x^2 y e^{-x^2 - y^2}(1 - 2y^2) \\ &= -2x^2 y(3 - 2y^2) e^{-x^2 - y^2} \end{aligned}$$

$$f_x = 0 \Rightarrow x = 0 \quad \text{or} \quad y = 0 \quad \text{or} \quad 1 - x^2 = 0 \\ \Rightarrow x = 1 \quad \text{or} \quad x = -1$$

$$f_y = 0 \Rightarrow x = 0 \quad \text{or} \quad 1 - 2y^2 = 0 \\ \Rightarrow y = \pm \frac{1}{\sqrt{2}}$$

$\rightarrow$ The critical points are $(0, y)$ for all y ,
 $(1, \frac{1}{\sqrt{2}})$, $(1, -\frac{1}{\sqrt{2}})$, $(-1, \frac{1}{\sqrt{2}})$, $(-1, -\frac{1}{\sqrt{2}})$.

$$\text{At } (0, y): D = \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = 0 \\ \Rightarrow \text{Second derivatives test tells us nothing.}$$

$f(x, y) > 0$	$f(x, y) > 0$	$(0, y)$ is	min	$\begin{cases} y > 0 \\ y = 0 \\ y < 0 \end{cases}$
			saddle	
$f(x, y) < 0$	$f(x, y) < 0$		max	

On the axes: $f(x, y) = 0$

$$A+ \left(1, \frac{1}{\sqrt{2}}\right) : D = 8e^{-3} > 0$$

$$f_{xx} = -2\sqrt{2} e^{-\frac{3}{2}} < 0$$

$$\Rightarrow \left(1, \frac{1}{\sqrt{2}}\right) \text{ is a local maximum.}$$

$$A+ \left(1, -\frac{1}{\sqrt{2}}\right) : D = 8e^{-3} > 0$$

$$f_{xx} = 2\sqrt{2} e^{-\frac{3}{2}} > 0$$

$$\Rightarrow \left(1, -\frac{1}{\sqrt{2}}\right) \text{ is a local minimum.}$$

$$A+ \left(-1, \frac{1}{\sqrt{2}}\right) : D = 8e^{-3} > 0$$

$$f_{xx} = -2\sqrt{2} e^{-\frac{3}{2}} < 0$$

$$\Rightarrow \left(-1, \frac{1}{\sqrt{2}}\right) \text{ is a local maximum.}$$

$$A+ \left(-1, -\frac{1}{\sqrt{2}}\right) : D = 8e^{-3} > 0$$

$$f_{xx} = 2\sqrt{2} e^{-\frac{3}{2}} > 0$$

$$\Rightarrow \left(-1, -\frac{1}{\sqrt{2}}\right) \text{ is a local minimum.}$$

2) $z^2 = xy + 1$, find point closest to origin.
 I.e. minimise $x^2 + y^2 + z^2 = x^2 + y^2 + xy + 1$.

$$\left. \begin{aligned} f_x(x,y) &= z x + y = 0 \Leftrightarrow y = -zx \\ f_y(x,y) &= z y + x = 0 \Leftrightarrow x = -zy \end{aligned} \right\} \begin{aligned} x &= 4x \\ \Rightarrow x &= 0 \\ \text{and } y &= 0. \end{aligned}$$

$$x=y=0 \Rightarrow z^2 = 1 \Rightarrow z = \pm 1$$

$$\Rightarrow \text{The points closest to the origin are } (0, 0, 1) \text{ and } (0, 0, -1).$$

3) Find $x, y, z > 0$ s.t. $x+y+z = 100$ and xyz is maximal.

$$x+y+z = 100 \rightarrow z = 100 - x - y$$

$$\Rightarrow \text{Want to maximise } f(x,y) = xy(100 - x - y)$$

$$f_x(x,y) = 100y - 2xy - y^2$$

$$f_y(x,y) = 100x - x^2 - 2xy$$

$$\begin{aligned} f_x = 0 &\Rightarrow y = 0 \quad \text{or} \quad 100 - 2x - y = 0 \\ f_y = 0 &\Rightarrow x = 0 \quad \text{or} \quad 100 - x - 2y = 0 \end{aligned}$$

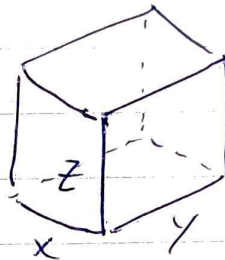
$$\begin{aligned} &\Rightarrow \begin{cases} 2x + y = 100 \\ x + 2y = 100 \end{cases} \Rightarrow \begin{cases} x - y = 0 \\ \Rightarrow x = y \end{cases} \\ &\Rightarrow 3x = 100 \end{aligned}$$

$$\Rightarrow x = y = \frac{100}{3}$$

$$\Rightarrow z = \frac{100}{3}$$

$$\Rightarrow xyz \text{ is maximal if } x = y = z = \frac{100}{3} \text{ (and then } xyz = \frac{1000000}{27} \text{)}$$

4) Find $x, y, z > 0$ s.t.
 $4(x + y + z) = c$ and
 xyz is maximal.



Similarly to problem 3, xyz is maximal
 if $x = y = z = \frac{c}{12}$.
 (and then $xyz = \frac{c^3}{1728}$).

5) $xyz = 32000$
 Minimise $xy + 2xz + 2yz$

$$z = \frac{32000}{xy}$$

$\Rightarrow$ Want to minimise $f(x, y) = xy + \frac{64000}{y} + \frac{64000}{x}$

$$\begin{aligned} f_x(x, y) &= y - \frac{64000}{x^2} = 0 \quad \text{if} \quad y = \frac{64000}{x^2} \\ f_y(x, y) &= x - \frac{64000}{y^2} = 0 \quad \text{if} \quad x = \frac{64000}{y^2} \end{aligned}$$

$$\Rightarrow x^2 y = x y^2 = 64000$$

Divide by $xy \Rightarrow x = y$

$$\Rightarrow x = \frac{64000}{x^2} \Rightarrow x^3 = 64000$$

$$\Rightarrow x = 40 \quad \text{and} \quad y = 40.$$

$$z = \frac{32000}{xy} = \frac{32000}{40^2} = 20$$

$$\Rightarrow (x, y, z) = (40, 40, 20)$$

$$6) L = \sqrt{x^2 + y^2 + z^2}, \text{ maximise } xyz.$$

Using Lagrange multipliers: let

$$f(x, y, z) = xyz, \quad g(x, y, z) = x^2 + y^2 + z^2$$

Maximise $f(x, y, z)$ subject to $g(x, y, z) = L^2$.

At the maximum, $\nabla f = \lambda \nabla g$ for some λ .

$$\nabla f = (yz, xz, xy)$$

$$\nabla g = (2x, 2y, 2z)$$

$$\Rightarrow \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} = \begin{pmatrix} 2\lambda x \\ 2\lambda y \\ 2\lambda z \end{pmatrix}$$

$$\lambda = \frac{yz}{2x} = \frac{xz}{2y} = \frac{xy}{2z}$$

$$\Rightarrow \cancel{xyz} \frac{xyz}{2\lambda} = x^2 = y^2 = z^2$$

$$\Rightarrow x = y = z = \frac{L}{\sqrt{3}}$$

$$\Rightarrow xyz = \frac{L^3}{3\sqrt{3}}$$

Question 7

$$i) \int_{-1}^3 \int_0^1 (1 + 4xy) dx dy =$$

$$\int_{-1}^3 [x + 2x^2 y]_{x=0}^{x=1} dy =$$

$$\int_{-1}^3 1 + 2y dy =$$

$$[y + y^2]_{y=-1}^{y=3} = 12 - 2 = 10$$

$$ii) \int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x} \right) dx dy =$$

$$\int_1^4 \left[\frac{x^2}{2y} + y \log x \right]_{x=1}^{x=2} dy =$$

$$\int_1^4 \frac{3}{2y} + y \log 2 dy =$$

$$\left[\frac{3}{2} \log y + \frac{y^2}{2} \log 2 \right]_{y=1}^{y=4} = \frac{3}{2} \log 4 + \frac{15}{2} \log 2$$

$$= \frac{21}{2} \log 2$$

Question 8

$$i) \iint_R \frac{xy^2}{x^2+1} dA, \text{ where } R = [0, 1] \times [-3, 3]$$

$$= \int_0^1 \int_{-3}^3 \frac{xy^2}{x^2+1} dy dx$$

$$= \int_0^1 \frac{x}{x^2+1} \left[\frac{y^3}{3} \right]_{y=-3}^{y=3} dx$$

$$= \int_0^1 \frac{18x}{x^2+1} dx$$

$$u = x^2 + 1$$

$$du = 2x dx$$

$$= \int_1^2 \frac{9}{u} du$$

$$= [9 \log u]_1^2 = 9 \log 2$$

2 ii) $\iint_R x \sin(x+y) dA$, $R = [0, \frac{\pi}{6}] \times [0, \frac{\pi}{3}]$

$$= \int_0^{\pi/3} \int_0^{\pi/6} x \sin(x+y) dx dy$$

$$= \int_0^{\pi/3} [-x \cos(x+y) + \sin(x+y)]_{x=0}^{x=\pi/6} dy$$

$$= \int_0^{\pi/3} -\frac{\pi}{6} \cos(\frac{\pi}{6}+y) + \sin(\frac{\pi}{6}+y) - \sin(y) dy$$

$$= \left[-\frac{\pi}{6} \sin(\frac{\pi}{6}+y) - \cos(\frac{\pi}{6}+y) + \cos(y) \right]_0^{\pi/3}$$

$$= -\frac{\pi}{6} - 0 + \frac{1}{2} - \left(-\frac{\pi}{6} \cdot \frac{1}{2} - \frac{1}{2}\sqrt{3} + 1 \right)$$

$$= \frac{-6 + 6\sqrt{3} - \pi}{12}$$

Question 9

1) $\int_{-2}^3 \int_0^1 z dx dy$ $3x + 2y + z = 12$
 $\Rightarrow z = 12 - 3x - 2y$

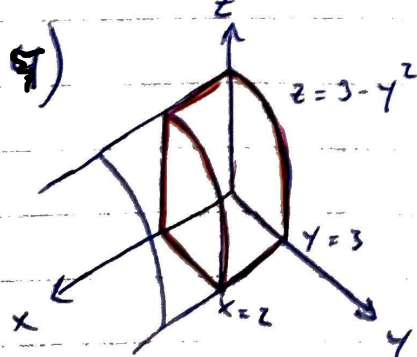
$$= \int_{-2}^3 \int_0^1 (12 - 3x - 2y) dx dy$$

$$= \int_{-2}^3 \left[12x - \frac{3}{2}x^2 - 2xy \right]_{x=0}^{x=1} dy$$

$$= \int_{-2}^3 \left(12 - \frac{3}{2} - 2y \right) dy = \int_{-2}^3 \left(\frac{21}{2} - 2y \right) dy$$

$$= \left[\frac{21}{2}y - y^2 \right]_{-2}^3 = \frac{95}{2}$$

Question 10



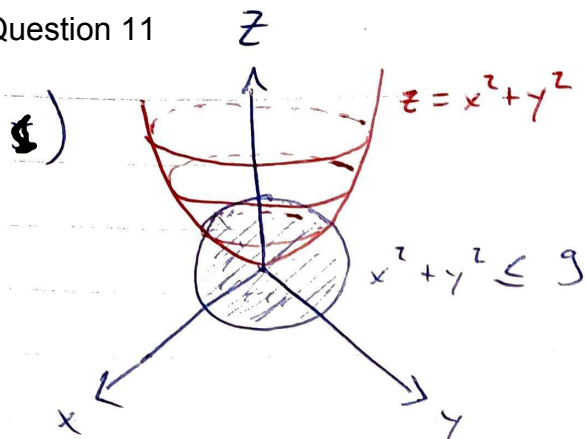
$$\int_0^3 \int_0^2 (3 - y^2) dx dy =$$

$$\int_0^3 [3x - xy^2]_{x=0}^{x=2} dy =$$

$$\int_0^3 (6 - 2y^2) dy =$$

$$\left[6y - \frac{2}{3}y^3 \right]_{y=0}^3 = 36$$

Question 11



$$\int_0^{2\pi} \int_0^3 z r dr d\theta =$$

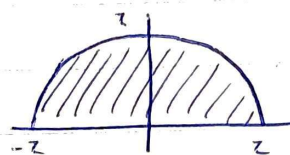
$$\int_0^{2\pi} \int_0^3 r^3 dr d\theta =$$

$$\int_0^{2\pi} \left[\frac{1}{4} r^4 \right]_{r=0}^3 d\theta =$$

$$\int_0^{2\pi} \frac{81}{4} d\theta = \frac{81\pi}{2}$$

Question 12

$$\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x^2 y^2 dx dy =$$



$$\int_0^{\pi} \int_0^2 r^2 \cos^2 \theta r^2 \sin^2 \theta r dr d\theta =$$

$$\int_0^{\pi} \left[\frac{r^6}{6} \cos^2 \theta \sin^2 \theta \right]_{r=0}^{r=2} d\theta =$$

$$\frac{64}{6} \int_0^{\pi} \cos^2 \theta \sin^2 \theta d\theta =$$

$$\frac{64}{6} \int_0^{\pi} \frac{(1 + \cos(2\theta))(1 - \cos(2\theta))}{4} d\theta =$$

$$\frac{16}{6} \int_0^{\pi} 1 - \cos^2 2\theta d\theta =$$

$$\frac{16}{6} \int_0^{\pi} 1 - \frac{1 + \cos(4\theta)}{2} d\theta =$$

$$\frac{16}{6} \left[\frac{1}{2} \theta - \frac{1}{8} \sin(4\theta) \right]_0^{\pi} = \frac{4\pi}{3}$$

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$$

$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}$$

Question 13

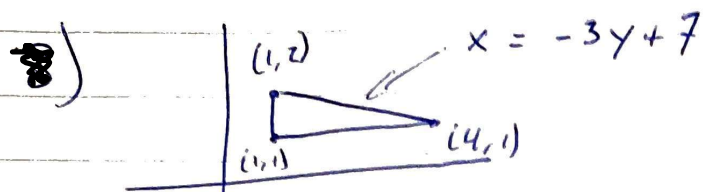
$$\int_0^1 \int_0^{x^2} x \cos y dy dx =$$

$$\int_0^1 [x \sin y]_{y=0}^{y=x^2} dx =$$

$$\int_0^1 x \sin x^2 dx = \begin{cases} u = x^2 \\ du = 2x dx \end{cases}$$

$$\int_0^1 \frac{1}{2} \sin u du = \left[-\frac{1}{2} \cos u \right]_0^1 = -\frac{1}{2} \cos 1 + \frac{1}{2} = \sin^2 \frac{1}{2}$$

Question 14



$$\int_1^2 \int_1^{-3y+7} xy \, dx \, dy =$$

$$\int_1^2 \left[\frac{1}{2} x^2 y \right]_{x=1}^{-3y+7} dy =$$

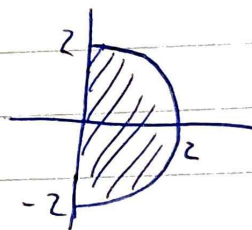
$$\int_1^2 \frac{1}{2} (9y^3 - 42y^2 + 48y) \, dy =$$

$$\left[\frac{9}{8} y^4 - 7y^3 + 12y^2 \right]_{y=1}^2 = \frac{31}{8}$$

Question 15

$$\iint_D e^{-x^2-y^2} \, dA =$$

$$\int_{-\pi/2}^{\pi/2} \int_0^2 e^{-r^2} r \, dr \, d\theta =$$



$$u = r^2$$

$$du = 2r \, dr$$

$$\int_{-\pi/2}^{\pi/2} \int_0^4 \frac{1}{2} e^{-u} \, du \, d\theta =$$

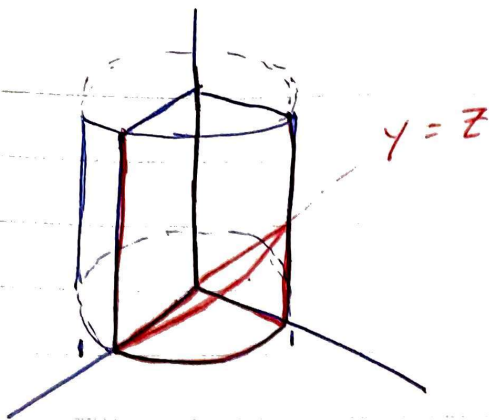
$$\int_{-\pi/2}^{\pi/2} \left[-\frac{1}{2} e^{-u} \right]_{u=0}^{u=4} d\theta =$$

$$\int_{-\pi/2}^{\pi/2} \left(-\frac{1}{2} e^{-4} + \frac{1}{2} \right) d\theta =$$

$$\left[\left(-\frac{1}{2} e^{-4} + \frac{1}{2} \right) \theta \right]_{\theta=-\pi/2}^{\pi/2} =$$

$$\frac{\pi}{2} (-e^{-4} + 1) = \frac{\pi}{2e^4} (e^4 - 1)$$

Question 16



$$\int_0^{\pi/2} \int_0^1 z r dr d\theta =$$

$$\int_0^{\pi/2} \int_0^1 r^2 \sin \theta dr d\theta =$$

$$\int_0^{\pi/2} \left[\frac{1}{3} r^3 \sin \theta \right]_{r=0}^{r=1} d\theta =$$

$$\int_0^{\pi/2} \frac{1}{3} \sin \theta d\theta =$$

$$\left[-\frac{1}{3} \cos \theta \right]_{\theta=0}^{\theta=\pi/2} = \frac{1}{3}$$