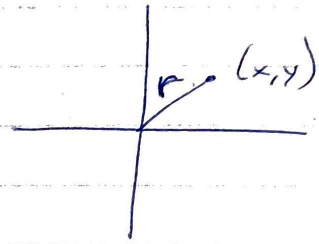
1 i) On the y-axis, $f(x,y) = f(0,y) = 0$.
Hence, $\lim_{y\to 0} f(0,y) = 0$.
On the curve $x = y^2$, $f(x,y) = \frac{y^4}{2y^4} = \frac{1}{2}$.
Hence, $\lim_{y\to 0} f(y^2,y) = \frac{1}{2}$.
Since these two limits are different,
$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ doesn't exist.

ii) For any point (x,y) let r be its distance to the origin.



Then $r^2 = x^2+y^2$,
$|x| \le r$ and $|y| \le r$.

Hence, $\left|\frac{3x^2y}{x^2+y^2}\right| \le \frac{3r^3}{r^2} = 3r$

$\Longrightarrow \left|\frac{3x^2y}{x^2+y^2}\right| \longrightarrow 0$ as $r \longrightarrow 0$
(i.e. as $(x,y) \to (0,0)$)

$\longrightarrow \lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$.

2 i) $f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$

$= \lim_{h\to 0} \frac{(x+h)y^2 - (x+h)^2y - xy^2 + x^2y}{h}$

$= \lim_{h\to 0} \frac{\cancel{xy^2} + hy^2 - \cancel{x^2y} - 2xhy - h^2y - \cancel{xy^2} + \cancel{x^2y}}{h}$

$= \lim_{h\to 0} y^2 - 2xy - hy = y^2 - 2xy$.

ii) $f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$

$= \lim_{h\to 0} \frac{x(y+h)^2 - x^2(y+h) - xy^2 + x^2y}{h}$

$= \lim_{h\to 0} \frac{\cancel{xy^2} + 2xyh + xh^2 - \cancel{x^2y} - x^2h - \cancel{xy^2} + \cancel{x^2y}}{h}$

$= \lim_{h\to 0} 2xy + xh - x^2 = 2xy - x^2$.

2 ii) $f_x(x,y) = \lim_{h\to 0} \dfrac{\frac{x+h}{x+h+y^2} - \frac{x}{x+y^2}}{h}$

$= \lim_{h\to 0} \dfrac{(x+h)(x+y^2) - x(x+h+y^2)}{(x+h+y^2)(x+y^2)h}$

$= \lim_{h\to 0} \dfrac{\cancel{x^2} \cancel{+xy^2} + \cancel{xh} + y^2h \cancel{-x^2} \cancel{-xh} \cancel{-xy^2}}{(x+h+y^2)(x+y^2)h}$

$= \lim_{h\to 0} \dfrac{y^2}{(x+h+y^2)(x+y^2)} = \dfrac{y^2}{(x+y^2)^2}$

$f_y(x,y) = \lim_{h\to 0} \dfrac{\frac{x}{x+(y+h)^2} - \frac{x}{x+y^2}}{h}$

$= \lim_{h\to 0} \dfrac{x(x+y^2) - x(x+(y+h)^2)}{(x+(y+h)^2)(x+y^2)h}$

$= \lim_{h\to 0} \dfrac{\cancel{x^2} \cancel{+xy^2} \cancel{-x^2} \cancel{-xy^2} - 2xyh - xh^2}{(x+(y+h)^2)(x+y^2)h}$

$= \lim_{h\to 0} \dfrac{-2xy - xh}{(x+(y+h)^2)(x+y^2)} = \dfrac{-2xy}{(x+y^2)^2}$

3 i) $\dfrac{\partial}{\partial x}(yz) = \dfrac{\partial}{\partial x} \ln(x+z)$

$y\dfrac{\partial z}{\partial x} = \dfrac{1}{x+z}\left(1 + \dfrac{\partial z}{\partial x}\right)$

$\left(y - \dfrac{1}{x+z}\right)\dfrac{\partial z}{\partial x} = \dfrac{1}{x+z}$

$\dfrac{\partial z}{\partial x} = \dfrac{\frac{1}{x+z}}{y - \frac{1}{x+z}} = \dfrac{1}{y(x+z) - 1}$

$\dfrac{\partial}{\partial y}(yz) = \dfrac{\partial}{\partial y} \ln(x+z)$

$z + y\dfrac{\partial z}{\partial y} = \dfrac{1}{x+z} \dfrac{\partial z}{\partial y}$

$\left(y - \dfrac{1}{x+z}\right)\dfrac{\partial z}{\partial y} = -z$

$\dfrac{\partial z}{\partial y} = \dfrac{-z}{y - \frac{1}{x+z}} = \dfrac{-z(x+z)}{y(x+z) - 1}$

3 ii) $\frac{\partial}{\partial x}(x - z) = \frac{\partial}{\partial x} \arctan(yz)$

$1 - \frac{\partial z}{\partial x} = \frac{1}{1+(yz)^2}\, y \frac{\partial z}{\partial x}$

$\left(1 + \frac{y}{1+(yz)^2}\right)\frac{\partial z}{\partial x} = 1$

$\frac{\partial z}{\partial x} = \frac{1}{1 + \frac{y}{1+(yz)^2}} = \frac{1+(yz)^2}{1+(yz)^2+y}$

$\frac{\partial}{\partial y}(x - z) = \frac{\partial}{\partial y} \arctan(yz)$

$-\frac{\partial z}{\partial y} = \frac{1}{1+(yz)^2}\left(z + y\frac{\partial z}{\partial y}\right)$

$\left(1 + \frac{y}{1+(yz)^2}\right)\frac{\partial z}{\partial y} = \frac{-z}{1+(yz)^2}$

$\frac{\partial z}{\partial y} = \frac{\frac{-z}{1+(yz)^2}}{1+\frac{y}{1+(yz)^2}} = \frac{-z}{1+(yz)^2+y}$

4 i) $f_x(x,y) = \sin(x+2y) + x\cos(x+2y)$

$f_{xy}(x,y) = 2\cos(x+2y) - 2x\sin(x+2y)$

Same.

$f_y(x,y) = 2x\cos(x+2y)$

$f_{yx}(x,y) = 2\cos(x+2y) - 2x\sin(x+2y)$

ii) $f_x(x,y) = 4x^3y^2 - 2y^5$

$f_{xy}(x,y) = 8x^3y - 10y^4$

Same

$f_y(x,y) = 2x^4y - 10xy^4$

$f_{yx}(x,y) = 8x^3y - 10y^4$

5 i) $f_x(x,y) = \frac{3}{3x+5y}$

$f_y(x,y) = \frac{5}{3x+5y}$

$f_{xx}(x,y) = \frac{-9}{(3x+5y)^2}$

$f_{xy}(x,y) = f_{yx}(x,y) = \frac{-15}{(3x+5y^2)}$

$f_{yy}(x,y) = \frac{-25}{(3x+5y)^2}$

5 ii) $z_x = \frac{x+y-x}{(x+y)^2} = \frac{y}{(x+y)^2}$

$z_y = \frac{-x}{(x+y)^2}$

$z_{xx} = \frac{-2y}{(x+y)^3}$

$z_{xy} = z_{yx} = \frac{(x+y)^2 - 2y(x+y)}{(x+y)^4} = \frac{x+y-2y}{(x+y)^3} = \frac{x-y}{(x+y)^3}$

$z_{yy} = \frac{2x}{(x+y)^3}$

6 i) $f_t = -cx^2e^{-ct}$

$f_{tt} = c^2x^2e^{-ct}$

$f_{ttt} = -c^3x^2e^{-ct}$

$f_{tx} = -2cxe^{-ct}$

$f_{txx} = -2ce^{-ct}$

ii) $f_x = -4\sin(4x+3y+2z)$

$f_{xy} = -12\cos(4x+3y+2z)$

$f_{xyz} = 24\sin(4x+3y+2z)$

$= f_{zyx}$

7) The ellipse is given by $4x^2 + 8 + z^2 = 16$

$\Longrightarrow 4x^2 + z^2 = 8.$

To find the slope of the tangent line, we use implicit differentiation:

$$8x + 2z\frac{dz}{dx} = 0$$

$$\frac{dz}{dx} = -\frac{8x}{2z} = -4\frac{x}{z}$$

$\Rightarrow$ The tangent line at the point $(1,2,2)$ has slope $-4\cdot\frac{1}{2} = -2$.

$\Rightarrow$ The equation of the tangent line is

$$z-2 = -2(x-1)$$

$$z = -2x+4.$$