1a) In the intersection points, $\begin{pmatrix} t \\ 0 \\ 2t-t^2 \end{pmatrix} = \begin{pmatrix} x \\ y \\ x^2+y^2 \end{pmatrix}$,

i.e. $x = t$, $y = 0$, $z = x^2+y^2 = t^2$,
but also $z = 2t - t^2$

$$\longrightarrow \quad t^2 = 2t - t^2$$
$$2t^2 = 2t$$
$$t = 0 \quad \text{or} \quad t = 1$$
$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

b) The sphere with centre (a,b,c) and radius r is given by
$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$
$$x^2 + y^2 + z^2 - 8x + 2y + 6z + 1 = 0$$
$$\Longrightarrow (x-4)^2 - 16 + (y+1)^2 - 1 + (z+3)^2 - 9 + 1 = 0$$
$$\Longrightarrow (x-4)^2 + (y+1)^2 + (z+3)^2 = 25$$
$$\longrightarrow \text{Centre} = (4, -1, -3)$$
$$\text{radius} = 5$$

2a) On the curve of intersection, we have
$y = x^2$ and $z = 4x^2 + y^2 = 4x^2 + x^4$
$\Longrightarrow$ If we set $x = t$, then $y = t^2$
and $z = 4t^2 + t^4$

$$\Longrightarrow t \longmapsto \begin{pmatrix} t \\ t^2 \\ 4t^2+t^4 \end{pmatrix}$$ is a parametrisation of the curve.

b) $F'(t) = 2t\,\underline{i} + 2t\,\underline{j} + \underline{k}$
$F(t) = -\underline{i} + \underline{j} + \underline{k} \longrightarrow t = 0$
$F'(0) = \underline{k}$
$\longrightarrow$ Tangent line at $-\underline{i} + \underline{j} + \underline{k}$ is given by
$$t \longmapsto t\,F'(0) + F(0)$$
$$= -\underline{i} + \underline{j} + (t+1)\,\underline{k}$$

3) The particles collide if $r_1(t) = r_2(t)$ for some $t \geq 0$.

$$\Rightarrow \quad t^2 = 4t - 3$$
$$7t - 12 = t^2$$
$$t^2 = 5t - 6$$

From first and third equation, we get

$$5t - 6 = 4t - 3$$
$$\Rightarrow t = 3$$

$\Rightarrow$ If the particles collide, they do so at $t=3$.

Check: $r_1(3) = 9\underline{i} + 9\underline{j} + 9\underline{k}$

$r_2(3) = 9\underline{i} + 9\underline{j} + 9\underline{k}$

$\Rightarrow$ They do collide.

4 a) Substitute the coordinates of $F(t)$ into the equation $4x + 5y - 2z = 18$:

$$4(2+3t) + 5(-4t) - 2(5+t) = 18$$
$$8 + 12t - 20t - 10 - 2t = 18$$
$$-10t = 20$$
$$t = -2$$

$\Rightarrow$ The point of intersection is

$$F(-2) = -4\underline{i} + 8\underline{j} + 3\underline{k}.$$

b) In general, the equation of a plane looks like

$$ax + by + cz = d.$$

Substituting the points $\underline{i}$, $2\underline{j}$ and $3\underline{k}$ gives

$a = d$

$2b = d$

$3c = d$

$\Rightarrow$ Taking $d = 6$, we get $a = 6$, $b = 3$, $c = 2$

$$\Rightarrow 6x + 3y + 2z = 6.$$

4 c) Consider the normal vectors

$$\underline{n}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \underline{n}_2 = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}.$$

The planes are perpendicular/parallel $\iff$ $\underline{n}_1$ and $\underline{n}_2$ are perpendicular/parallel.

$\underline{n}_1$ and $\underline{n}_2$ are not parallel $\Rightarrow$ the planes are not parallel.

$$\underline{n}_1 \cdot \underline{n}_2 = 2 - 3 - 4 = -5 \neq 0$$

$\Longrightarrow$ not perpendicular.

If θ is the angle between the planes (= the angle between $\underline{n}_1$ and $\underline{n}_2$), then

$$\cos\theta = \frac{\underline{n}_1 \cdot \underline{n}_2}{\|\underline{n}_1\| \, \|\underline{n}_2\|}.$$

$$\|\underline{n}_1\| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}, \quad \|\underline{n}_2\| = \sqrt{2^2 + (-3)^2 + 4^2} = \sqrt{29}$$

$$\Longrightarrow \cos\theta = \frac{-5}{\sqrt{3} \cdot \sqrt{29}}$$

$$\Longrightarrow \cos\theta \approx 2.137 \text{ rad} \approx 122.4^\circ$$

d) Any plane parallel to $x + 4y - 3z = 1$ has an equation of the form $x + 4y - 3z = d$.
To find d, we substitute the point $2\underline{i} + \underline{j}$:

$$2 + 4 = d \Longrightarrow d = 6$$

$$\Longrightarrow x + 4y - 3z = 6.$$

5a) $F'(t) = 2t\,\underline{i} + 5\,\underline{j} + (12t - 16)\,\underline{k}$

$\|F'(t)\| = \sqrt{(2t)^2 + 5^2 + (12t - 16)^2}$

$= \sqrt{8t^2 - 64t + 281}$

Want to minimise $\sqrt{8t^2 - 64t + 281}$,
or equivalently, minimise $8t^2 - 64t + 281$.

$\frac{d}{dt}(8t^2 - 64t + 281) = 16t - 64 = 0$
$\Longrightarrow t = 4$ is the time when the speed is the least.

b) Let $\underline{v}(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{pmatrix}$ be the velocity vector and let c be the (constant) speed.

$$\sqrt{v_1(t)^2 + v_2(t)^2 + v_3(t)^2} = c$$
$$v_1(t)^2 + v_2(t)^2 + v_3(t)^2 = c^2$$

Differentiate:
$$2v_1(t)v_1'(t) + 2v_2(t)v_2'(t) + 2v_3(t)v_3'(t) = 0.$$
Note that the left-hand side is equal to
$$2\,\underline{v}(t) \cdot \underline{v}'(t)$$
Hence, $\underline{v}(t) \cdot \underline{v}'(t) = 0$, i.e. the velocity and acceleration vectors are perpendicular.

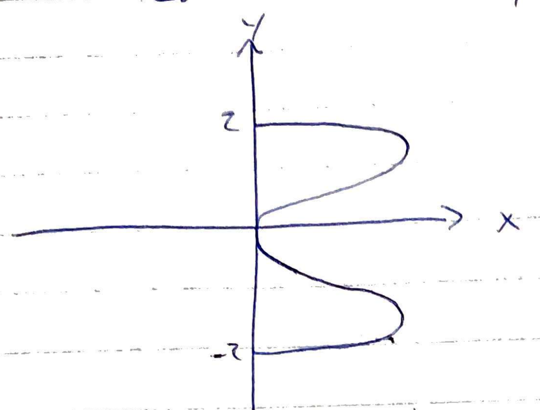
6) We can find a parametrisation of the ellipsoid by taking a parametrisation of the unit sphere and multiplying by the half-axes.
The sphere is parametrised by

$$(\phi, \theta) \longmapsto \begin{pmatrix} \sin\phi \cos\theta \\ \sin\phi \sin\theta \\ \cos\phi \end{pmatrix}$$

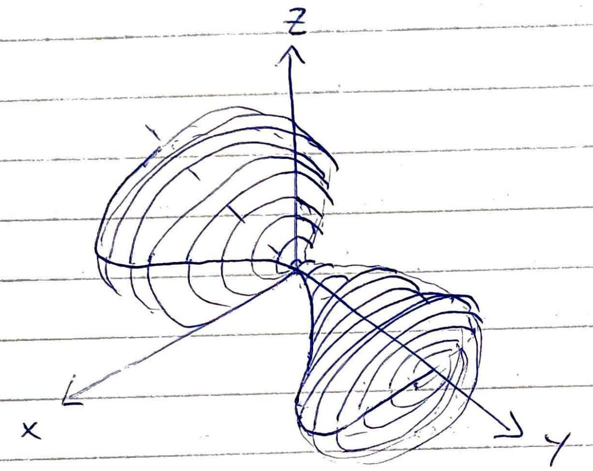
Hence, the ellipsoid is parametrised by

$$(\phi, \theta) \longmapsto \begin{pmatrix} 2\sin\phi \cos\theta \\ \sin\phi \sin\theta \\ 3\cos\phi \end{pmatrix}.$$

7) The curve $x = 4y^2 - y^4$ looks like



If we rotate this about the y-axis, we get



Fix $y = t$. The part of the surface where $y = t$ is a circle with radius $4t^2 - t^4$, so it's parametrised by

$$\theta \longmapsto ((4t^2 - t^4)\cos\theta, (4t^2 - t^4)\sin\theta).$$

Hence, the surface of revolution is parametrised by

$$(t, \theta) \longmapsto \begin{pmatrix} (4t^2 - t^4)\cos\theta \\ t \\ (4t^2 - t^4)\sin\theta \end{pmatrix}$$