BAILEY PAIRS, EICHLER INTEGRALS AND UNIFIED WITTEN-RESHETIKHIN-TURAEV INVARIANTS

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Abstract. In 1999, Lawrence and Zagier expressed the Witten-Reshetikhin-Turaev (WRT) invariant of the Poincaré homology sphere as the limiting value of the Eichler integral of a weight 3/2 modular form. Habiro's construction of the unified WRT invariant subsequently recast this result as an identity for a q-hypergeometric series at roots of unity. This motivated Hikami to prove analogous q-series identities involving the unified WRT invariants of certain Brieskorn homology spheres. Hikami also made several conjectures of a similar type for q-series with no apparent connection to quantum invariants. In this paper we use the Bailey pair machinery and a novel relation between incomplete quadratic Gauss sums with periodic coefficients to construct infinite families of identities between q-multisums at roots of unity and limiting values of Eichler integrals of weight 3/2 modular forms. These identities include all of Hikami's results and conjectures as well as a generalization of the result of Lawrence and Zagier.

1. Introduction

In 1999, Lawrence and Zagier made a prescient observation connecting quantum invariants and modular forms [23]. To state their result, let $Z_\zeta(\mathcal{M})$ denote the Witten-Reshetikhin-Turaev (WRT) invariant of a closed and oriented 3-manifold $\mathcal{M}$ at the root of unity ζ. Next let $\boldsymbol{p} = (p_1, p_2, p_3)$ be a triple of pairwise coprime positive integers and $P := p_1p_2p_3$. For any triple $\boldsymbol{\ell} = (\ell_1, \ell_2, \ell_3) \in \mathbb{Z}^3$ satisfying $0 < \ell_j < p_j$, define the odd periodic function

$$\chi_{\boldsymbol{p}}^{\boldsymbol{\ell}}(n) = \begin{cases} -\epsilon_1\epsilon_2\epsilon_3 & \text{if } n \equiv P\left(1 + \displaystyle\sum_{j=1}^{3} \frac{\epsilon_j \ell_j}{p_j}\right) \pmod{2P}, \\ 0 & \text{otherwise} \end{cases}$$

where $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_3) \in \{\pm 1\}^3$. For $q = e^{2\pi i \tau}$, $\tau \in \mathbb{H}$, consider the Eichler integral

$$\widetilde{\Phi}_{\boldsymbol{p}}^{\boldsymbol{\ell}}(\tau) := \sum_{n=0}^{\infty} \chi_{\boldsymbol{p}}^{\boldsymbol{\ell}}(n) q^{\frac{n^2}{4P}} \tag{1.1}$$

of the weight 3/2 modular form

$$\Phi_{\boldsymbol{p}}^{\boldsymbol{\ell}}(\tau) := \sum_{n=0}^{\infty} n \chi_{\boldsymbol{p}}^{\boldsymbol{\ell}}(n) q^{\frac{n^2}{4P}}.$$

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Their main result [23, Theorem 1] reads as follows. For coprime positive integers M and N, let $\zeta_N^M := e^{\frac{2\pi i M}{N}}$. Then for the Poincaré homology sphere $\Sigma(2,3,5)$ we have

$$1 + \zeta_N^M (1 - \zeta_N^M) Z_{\zeta_N^M}(\Sigma(2,3,5)) = -\frac{1}{2} \lim_{\tau \to \frac{M}{N}} q^{-\frac{1}{120}} \widetilde{\Phi}_{(2,3,5)}^{(1,1,1)}(\tau). \tag{1.2}$$

They also established that the right-hand side of (1.2) has "a modular property modulo smooth functions", a notion which would later become the defining attribute of Zagier's quantum modular forms [36]. Any Eichler integral defined by (1.1) is now known to be an example of a quantum modular form [14, Section 4.4].

Following up on the work of Lawrence and Zagier, Hikami wrote a series of papers [16–19] wherein he showed that limiting values of Eichler integrals give the WRT invariants for several families of 3-manifolds, including the Brieskorn homology spheres. As an application, he used the modularity to find asymptotic expansions for $Z_{\zeta_N}(\mathcal{M})$ as $N \to \infty$. Hikami's results have recently been used in the study of limiting values of an SU(2) WRT function [2,3,9] and towards progress on a quantum modularity conjecture for WRT invariants [30, Theorem 1].

The identity (1.2) may be recast as an identity for q-hypergeometric series at roots of unity using Habiro's unified WRT invariant. Habiro [15] showed that if $\mathcal{M}$ is an integral homology sphere, there is a q-hypergeometric series $I(\mathcal{M}; q)$ such that for any root of unity ζ one has

$$I(\mathcal{M}; \zeta) = Z_\zeta(\mathcal{M}).$$

In the case of the Poincaré homology sphere, the unified WRT invariant is [24]

$$I(\Sigma(2,3,5); q) = \frac{1}{1-q} \sum_{n \geq 0} q^n (q^{n+1})_{n+1},$$

where

$$(a)_n = (a; q)_n := \prod_{k=1}^{n} (1 - aq^{k-1}).$$

Thus (1.2) says that

$$H(\zeta_N^M) = -\frac{1}{2} \lim_{\tau \to \frac{M}{N}} q^{-\frac{1}{120}} \widetilde{\Phi}_{(2,3,5)}^{(1,1,1)}(\tau), \tag{1.3}$$

where

$$H(q) := \sum_{n \geq 0} q^n (q^n)_n.$$

Explicit expressions for $I(\mathcal{M}; q)$ are uncommon as they require knowledge of the cyclotomic expansion of the colored Jones polynomial of the knot K from which $\mathcal{M}$ is constructed using Dehn surgery. One case where this has been fully carried out is for the Brieskorn homology spheres $\Sigma(2,3,6p-1)$ and $\Sigma(2,3,6p+1)$, which arise from (-1)-surgery and $(+1)$-surgery, respectively, along the twist knot K_p, $p > 0$ [21, Theorem C.1]. Specifically, Hikami [21] showed that one has

$$(1-q) I(\Sigma(2,3,6p-1); q) = H_p^{(1)}(q) \tag{1.4}$$

for $p > 1$ and

$$(1-q) I(\Sigma(2,3,6p+1); q) = H_p^{(8)}(q) \tag{1.5}$$

for $p \geq 1$ where[1]

$$H_p^{(1)}(q) := \sum_{n_p \geq \cdots \geq n_1 \geq 0} q^{n_p} (q^{n_p+1})_{n_p+1} \prod_{i=1}^{p-1} q^{n_i(n_i+1)} \begin{bmatrix} n_{i+1} \\ n_i \end{bmatrix} \tag{1.6}$$

and

$$H_p^{(8)}(q) := \sum_{n_p \geq \cdots \geq n_1 \geq 0} q^{-n_p(n_p+2)} (q^{n_p+1})_{n_p+1} \prod_{i=1}^{p-1} q^{n_i(n_i+1)} \begin{bmatrix} n_{i+1} \\ n_i \end{bmatrix}. \tag{1.7}$$

Here

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{(q)_n}{(q)_{n-k}(q)_k} \tag{1.8}$$

is the q-binomial coefficient. Combining (1.4) and (1.5) with expressions for the WRT invariants of the Brieskorn homology spheres in terms of Eichler integrals [16], Hikami deduced the identities [21, Theorem 3.3]

$$H_p^{(1)}(\zeta_N) = (1-\zeta_N) I(\Sigma(2,3,6p-1); \zeta_N) = -\frac{1}{2} \lim_{\tau \to \frac{1}{N}} q^{-\frac{(6p+5)^2}{24(6p-1)}} \widetilde{\Phi}^{(1,1,1)}_{(2,3,6p-1)}(\tau) \tag{1.9}$$

for $p > 1$ and [21, Theorem 2.4 and Proposition 3.2]

$$H_p^{(8)}(\zeta_N) = (1-\zeta_N) I(\Sigma(2,3,6p+1); \zeta_N) = -\frac{1}{2} \lim_{\tau \to \frac{1}{N}} q^{1-\frac{(6p-5)^2}{24(6p+1)}} \widetilde{\Phi}^{(1,1,1)}_{(2,3,6p+1)}(\tau) \tag{1.10}$$

for $p \geq 1$.

In [21], Hikami conjectured a number of identities for q-series at roots of unity which closely resemble (1.3), (1.9) and (1.10) but for which no interpretation in terms of quantum invariants is known. These conjectures are also presented in [29] and two more were made in private communication [22]. To state them, we define the following families of q-series. For $p \geq 1$, let

$$H_p^{(2)}(q) := \sum_{n_p \geq \cdots \geq n_1 \geq 0} q^{n_p} (q^{n_p})_{n_p} \prod_{i=1}^{p-1} q^{n_i^2} \begin{bmatrix} n_{i+1} \\ n_i \end{bmatrix}, \tag{1.11}$$

$$H_p^{(3)}(q) := \sum_{n_p \geq \cdots \geq n_1 \geq 0} q^{2n_p} (q^{n_p+1})_{n_p} \prod_{i=1}^{p-1} q^{n_i(n_i+1)} \begin{bmatrix} n_{i+1} \\ n_i \end{bmatrix}, \tag{1.12}$$

$$H_p^{(4)}(q) := \sum_{n_p \geq \cdots \geq n_1 \geq 0} q^{n_p} (q^{n_p+1})_{n_p} \prod_{i=1}^{p-1} q^{n_i(n_i+1)} \begin{bmatrix} n_{i+1} \\ n_i \end{bmatrix}, \tag{1.13}$$

$$H_p^{(5)}(q) := \sum_{n_p \geq \cdots \geq n_1 \geq 0} q^{n_p} (q^{n_p+1})_{n_p} \prod_{i=1}^{p-1} q^{n_i^2} \begin{bmatrix} n_{i+1} \\ n_i \end{bmatrix}, \tag{1.14}$$

$$H_p^{(6)}(q) := \sum_{n_p \geq \cdots \geq n_1 \geq 0} q^{-n_p^2} (q^{n_p+1})_{n_p} \prod_{i=1}^{p-1} q^{n_i^2+n_i} \begin{bmatrix} n_{i+1} \\ n_i \end{bmatrix} \tag{1.15}$$

[1]Here and throughout, we follow the notation $H_p^{(i)}(q)$ for the q-multisums (1.6) and (1.11)–(1.14) as given in [29]. We have labelled (1.7), (1.15), (1.16), (1.24) and (1.25) accordingly.

and

$$H_p^{(7)}(q) := \sum_{n_p \geq \cdots \geq n_1 \geq 0} q^{-n_p^2 - n_p} (q^{n_p+1})_{n_p} \prod_{i=1}^{p-1} q^{n_i^2 + n_i} \begin{bmatrix} n_{i+1} \\ n_i \end{bmatrix}. \tag{1.16}$$

In [21, (3.23), (3.25b)–(3.25d), (3.35)] and [22], the following conjectures were made.

Conjecture 1.1. *We have*

$$H_2^{(2)}(\zeta_N) = -\frac{1}{2} \lim_{\tau \to \frac{1}{N}} q^{-\frac{1}{264}} \widetilde{\Phi}_{(2,3,11)}^{(1,1,2)}(\tau), \tag{1.17}$$

$$H_2^{(3)}(\zeta_N) = -\frac{1}{2} \lim_{\tau \to \frac{1}{N}} q^{-(1+\frac{49}{264})} \widetilde{\Phi}_{(2,3,11)}^{(1,1,3)}(\tau), \tag{1.18}$$

$$H_1^{(4)}(\zeta_N) = -\frac{1}{2} \lim_{\tau \to \frac{1}{N}} q^{-\frac{49}{120}} \widetilde{\Phi}_{(2,3,5)}^{(1,1,2)}(\tau), \tag{1.19}$$

$$H_2^{(4)}(\zeta_N) = -\frac{1}{2} \lim_{\tau \to \frac{1}{N}} q^{-\frac{169}{264}} \widetilde{\Phi}_{(2,3,11)}^{(1,1,4)}(\tau), \tag{1.20}$$

$$H_p^{(5)}(\zeta_N) = -\frac{1}{2} \lim_{\tau \to \frac{1}{N}} q^{\frac{1+48p}{24(1-6p)}} \widetilde{\Phi}_{(2,3,6p-1)}^{(1,1,3p-1)}(\tau) \tag{1.21}$$

where $p \geq 1$,

$$H_1^{(6)}(\zeta_N) = -\frac{1}{2} \lim_{\tau \to \frac{1}{N}} q^{-\frac{25}{168}} \widetilde{\Phi}_{(2,3,7)}^{(1,1,2)}(\tau) \tag{1.22}$$

and

$$H_1^{(7)}(\zeta_N) = -\frac{1}{2} \lim_{\tau \to \frac{1}{N}} q^{\frac{47}{168}} \widetilde{\Phi}_{(2,3,7)}^{(1,1,3)}(\tau). \tag{1.23}$$

The purpose of this paper is to use the Bailey pair machinery and a novel relation between incomplete quadratic Gauss sums with periodic coefficients to construct identities between each of the infinite families in (1.11)–(1.16) evaluated at roots of unity and limiting values of Eichler integrals of weight 3/2 modular forms. This gives new proofs of (1.3), (1.9) and (1.10) as well as a proof and generalization of Conjecture 1.1.

The Bailey pair method also leads to the discovery of further identities between q-series at roots of unity and Eichler integrals. We will give two examples which involve the series

$$H_p^{(9)}(q) := \sum_{n_p \geq \cdots \geq n_1 \geq 0} q^{-n_p^2 - n_p} (q^{n_p+1})_{n_p} \prod_{i=1}^{p-1} q^{n_i^2} \begin{bmatrix} n_{i+1} \\ n_i \end{bmatrix} \tag{1.24}$$

and

$$H_p^{(10)}(q) := \sum_{n_p \geq \cdots \geq n_1 \geq 0} q^{-n_p^2} (q^{n_p})_{n_p} \prod_{i=1}^{p-1} q^{n_i^2} \begin{bmatrix} n_{i+1} \\ n_i \end{bmatrix}, \tag{1.25}$$

defined for $p \geq 1$. We now state our main result.

Theorem 1.2. *Let M and N be coprime positive integers. For $p \geq 2$, we have*

$$H_p^{(1)}(\zeta_N^M) = -\frac{1}{2} \lim_{\tau \to \frac{M}{N}} q^{-\frac{(6p+5)^2}{24(6p-1)}} \widetilde{\Phi}_{(2,3,6p-1)}^{(1,1,1)}(\tau) \tag{1.26}$$

while for $p = 1$

$$H_1^{(1)}(\zeta_N^M) = -\zeta_N^{-M} - \frac{1}{2} \lim_{\tau \to \frac{M}{N}} q^{-\frac{121}{120}} \widetilde{\Phi}_{(2,3,5)}^{(1,1,1)}(\tau). \tag{1.27}$$

For $p \geq 1$, we have

$$H_p^{(2)}(\zeta_N^M) = -\frac{1}{2} \lim_{\tau \to \frac{M}{N}} q^{-\frac{1}{24(6p-1)}} \widetilde{\Phi}_{(2,3,6p-1)}^{(1,1,p)}(\tau), \tag{1.28}$$

$$H_p^{(3)}(\zeta_N^M) = -\frac{1}{2} \lim_{\tau \to \frac{M}{N}} q^{-1-\frac{(6p-5)^2}{24(6p-1)}} \widetilde{\Phi}_{(2,3,6p-1)}^{(1,1,2p-1)}(\tau), \tag{1.29}$$

$$H_p^{(4)}(\zeta_N^M) = -\frac{1}{2} \lim_{\tau \to \frac{M}{N}} q^{-\frac{(6p+1)^2}{24(6p-1)}} \widetilde{\Phi}_{(2,3,6p-1)}^{(1,1,2p)}(\tau), \tag{1.30}$$

$$H_p^{(5)}(\zeta_N^M) = -\frac{1}{2} \lim_{\tau \to \frac{M}{N}} q^{p-1-\frac{(12p-5)^2}{24(6p-1)}} \widetilde{\Phi}_{(2,3,6p-1)}^{(1,1,3p-1)}(\tau), \tag{1.31}$$

$$H_p^{(6)}(\zeta_N^M) = -\frac{1}{2} \lim_{\tau \to \frac{M}{N}} q^{-\frac{(6p-1)^2}{24(6p+1)}} \widetilde{\Phi}_{(2,3,6p+1)}^{(1,1,2p)}(\tau), \tag{1.32}$$

$$H_p^{(7)}(\zeta_N^M) = -\frac{1}{2} \lim_{\tau \to \frac{M}{N}} q^{1-\frac{(6p+5)^2}{24(6p+1)}} \widetilde{\Phi}_{(2,3,6p+1)}^{(1,1,2p+1)}(\tau), \tag{1.33}$$

$$H_p^{(8)}(\zeta_N^M) = -\frac{1}{2} \lim_{\tau \to \frac{M}{N}} q^{1-\frac{(6p-5)^2}{24(6p+1)}} \widetilde{\Phi}_{(2,3,6p+1)}^{(1,1,1)}(\tau), \tag{1.34}$$

$$H_p^{(9)}(\zeta_N^M) = -\frac{1}{2} \lim_{\tau \to \frac{M}{N}} q^{p-1-\frac{(12p-1)^2}{24(6p+1)}} \widetilde{\Phi}_{(2,3,6p+1)}^{(1,1,3p)}(\tau) \tag{1.35}$$

and

$$H_p^{(10)}(\zeta_N^M) = -\frac{1}{2} \lim_{\tau \to \frac{M}{N}} q^{-\frac{1}{24(6p+1)}} \widetilde{\Phi}_{(2,3,6p+1)}^{(1,1,p)}(\tau). \tag{1.36}$$

Remark 1.3. Observe that the $p = 1$ case of (1.28) recovers (1.3) while the $M = 1$ cases of (1.26) and (1.34) yield (1.9) and (1.10), respectively. We have also included (1.27) for completeness (cf. [21, (3.8), (3.10)]). Note that Theorem 1.2 embeds Hikami's conjectures for one single value of p (with the exception of (1.21)) into an infinite family of identities. Finally, our main result and [14, Section 4.4] imply that each $H_p^{(i)}(q)$ is a quantum modular form.

The paper is organized as follows. In Section 2, we recall the necessary background on the Bailey pair machinery and express all of the series $H_p^{(i)}(q)$ as finite polynomial sums at roots of unity. See Theorems 2.7–2.16. An elucidatory principle in this section arises from the broader perspective of quantum q-series identities [26, 27], a recent development whose applications include new proofs for identities at roots of unity for Ramanujan's σ-function and its companion $\sigma^*(q)$ [5, 7], a "duality" result for the Kontsevich-Zagier strange function [6, 10] and Zagier's strange identity and Hikami's generalization thereof [20, 35]. For further work in this direction, see [8, 11, 12, 28]. In Section 3, we prove key properties for and relations between incomplete quadratic Gauss sums. In Section 4, we convert the finite polynomial sums in Section 2 to limiting values of Eichler integrals, thereby proving Theorem 1.2.

2. Bailey pairs and key identities

2.1. **Preliminaries and auxiliary results.** Recall that a Bailey pair relative to a is a pair of sequences $(\alpha_n, \beta_n)_{n\geq 0}$ satisfying

$$\beta_n = \sum_{k=0}^{n} \frac{\alpha_k}{(q)_{n-k}(aq)_{n+k}} \tag{2.1}$$

$$= \frac{1}{(q)_n(aq)_n} \sum_{k=0}^{n} \frac{(q^{-n})_k}{(aq^{n+1})_k} (-1)^k q^{nk-\binom{k}{2}} \alpha_k. \tag{2.2}$$

The first equation is the original definition [4], while the second follows from an application of the identity [13, Appendix I, Eq. (I.10)]

$$(x)_{n-k} = \frac{(x)_n}{(q^{1-n}/x)_k} (-q/x)^k q^{\binom{k}{2}-nk}. \tag{2.3}$$

We begin by reviewing two important facts about Bailey pairs. First, we require a special case of the Bailey lemma [4].

Lemma 2.1. *If (α_n, β_n) is a Bailey pair relative to a, then so is (α'_n, β'_n), where*

$$\alpha'_n = a^n q^{n^2} \alpha_n \tag{2.4}$$

and

$$\beta'_n = \sum_{j=0}^{n} \frac{a^j q^{j^2}}{(q)_{n-j}} \beta_j. \tag{2.5}$$

Second, we have a special case of the Bailey lattice [1].

Lemma 2.2. *If (α_n, β_n) is a Bailey pair relative to q^2, then (α^*_n, β_n) is a Bailey pair relative to q, where*

$$\alpha^*_n = (1-q^2)\left(\frac{\alpha_n}{1-q^{2n+2}} - \frac{q^{2n}\alpha_{n-1}}{1-q^{2n}}\right). \tag{2.6}$$

By convention we take $\alpha_{-1} = 0$.

Using Lemmas 2.1 and 2.2, we establish three auxiliary results which will allow us to relate the q-multisums in Theorem 1.2 to polynomial sums at roots of unity. The first result is for Bailey pairs relative to q.

Proposition 2.3. *If (α_k, β_k) is a Bailey pair relative to q, then for any $p \geq 1$ we have*

$$\sum_{n-1\geq n_p\geq \cdots \geq n_1\geq 0} \frac{q^{(n-n_p-1)(n-n_p)}\beta_{n-n_p-1}}{(q)_{n_p}} \prod_{i=1}^{p-1} q^{(n-n_i-1)(n-n_i)} \begin{bmatrix} n_{i+1} \\ n_i \end{bmatrix}$$
$$= \frac{1}{(q)_{n-1}(q^2)_{n-1}} \sum_{k=0}^{n-1} \frac{(q^{1-n})_k}{(q^{1+n})_k} (-1)^k q^{nk-\binom{k+1}{2}+p(k^2+k)} \alpha_k. \tag{2.7}$$

Proof. Suppose that (α_k, β_k) is a Bailey pair relative to q. Iterating (2.4) and (2.5) p times we obtain another Bailey pair relative to q,

$$\alpha_k^{(p)} = q^{p(k^2+k)}\alpha_k \tag{2.8}$$

and

$$\beta_k^{(p)} = \sum_{k \geq n_p \geq \cdots \geq n_1 \geq 0} \frac{q^{n_p(n_p+1)+\cdots+n_1(n_1+1)}}{(q)_{k-n_p}\cdots(q)_{n_2-n_1}}\beta_{n_1}. \tag{2.9}$$

Now in $\beta_k^{(p)}$ we replace k by $n-1$ and then make the change of indices $n_i \to n-1-n_{p-i+1}$ for $1 \leq i \leq p$. The result is

$$\beta_{n-1}^{(p)} = \sum_{n-1 \geq n_p \geq \cdots \geq n_1 \geq 0} \frac{q^{(n-n_1-1)(n-n_1)+\cdots+(n-n_p-1)(n-n_p)}}{(q)_{n_1}(q)_{n_2-n_1}\cdots(q)_{n_p-n_{p-1}}}\beta_{n-n_p-1}. \tag{2.10}$$

Multiplying numerator and denominator by $(q)_{n_p}$ and then converting to q-binomial coefficients (1.8) gives the left-hand side of (2.7). The right-hand side follows from inserting (2.8) into (2.2). This completes the proof. □

Our second auxiliary result is analogous to Proposition 2.3, but for Bailey pairs relative to q^2. In what follows, we define a_n^* for any sequence a_n by

$$a_n^* = (1-q^2)\left(\frac{a_n}{1-q^{2n+2}} - \frac{q^{2n}a_{n-1}}{1-q^{2n}}\right) \tag{2.11}$$

in analogy with (2.6).

Proposition 2.4. *If (α_k, β_k) is a Bailey pair relative to q^2, then for any $p \geq 1$ we have*

$$\begin{aligned} \sum_{n-1 \geq n_p \geq \cdots \geq n_1 \geq 0} & \frac{q^{(n-n_p-1)(n-n_p+1)}\beta_{n-n_p-1}}{(q)_{n_p}} \prod_{i=1}^{p-1} q^{(n-n_i-1)(n-n_i+1)} \begin{bmatrix} n_{i+1} \\ n_i \end{bmatrix} \\ &= \frac{1}{(q)_{n-1}(q^2)_{n-1}} \sum_{k=0}^{n-1} \frac{(q^{1-n})_k}{(q^{1+n})_k}(-1)^k q^{nk-\binom{k+1}{2}}(q^{p(k^2+2k)}\alpha_k)^*. \end{aligned} \tag{2.12}$$

Proof. Suppose that (α_k, β_k) is a Bailey pair relative to q^2. We argue as in Proposition 2.3 with the same iterations of (2.4) and (2.5) (but with $a = q^2$) and the same changes of indices. The result is a Bailey pair relative to q^2,

$$\alpha_k^{(p)} = q^{p(k^2+2k)}\alpha_k \tag{2.13}$$

and

$$\beta_{n-1}^{(p)} = \sum_{n-1 \geq n_p \geq \cdots \geq n_1 \geq 0} \frac{q^{(n-n_1-1)(n-n_1+1)+\cdots+(n-n_p-1)(n-n_p+1)}}{(q)_{n_1}(q)_{n_2-n_1}\cdots(q)_{n_p-n_{p-1}}}\beta_{n-n_p+1}. \tag{2.14}$$

Now we change this to a Bailey pair relative to q using Lemma 2.2 and then insert it into (2.2) to complete the proof. □

The third auxiliary result also applies to Bailey pairs relative to q^2, but in a different way.

Proposition 2.5. *If (α_k, β_k) is a Bailey pair relative to q^2, then for any $p \geq 1$ we have*

$$\sum_{n-1 \geq n_p \geq \cdots \geq n_1 \geq 0} \frac{q^{(n-n_p-1)(n-n_p+1)}\beta_{n-n_p-1}}{(q)_{n_p}} \prod_{i=1}^{p-1} q^{(n-n_i-1)(n-n_i+1)} \begin{bmatrix} n_{i+1} \\ n_i \end{bmatrix}$$
$$= \frac{(1-q^2)}{(q)_{n-1}(q)_n} \sum_{k=0}^{n-1} \frac{(q^{1-n})_k}{(q^{2+n})_k} (-1)^k q^{nk-\binom{k+1}{2}+p(k^2+2k)} \alpha_k. \tag{2.15}$$

Proof. This follows from substituting (2.13) and (2.14) into the definition of a Bailey pair in (2.2) with $a = q^2$. □

2.2. **An arsenal of Bailey pairs.** Here we collect all of the Bailey pairs we will use in the proof of Theorem 1.2. There are eight Bailey pairs that we cite from the literature – six Bailey pairs relative to q and two Bailey pairs relative to q^2. First, we have four Bailey pairs relative to q from Slater's list [33, p.463, A(2), A(4), A(6), A(8)]:

$$\alpha_k = \begin{cases} q^{6r^2-r} & \text{if } k = 3r-1, \\ q^{6r^2+r} & \text{if } k = 3r, \\ -q^{6r^2+5r+1} - q^{6r^2+7r+2} & \text{if } k = 3r+1 \end{cases} \quad \text{and} \quad \beta_k = \frac{1}{(q^2)_{2k}}, \tag{2.16}$$

$$\alpha_k = \begin{cases} q^{6r^2-4r} & \text{if } k = 3r-1, \\ q^{6r^2+4r} & \text{if } k = 3r, \\ -q^{6r^2+8r+2} - q^{6r^2+4r} & \text{if } k = 3r+1 \end{cases} \quad \text{and} \quad \beta_k = \frac{q^k}{(q^2)_{2k}}, \tag{2.17}$$

$$\alpha_k = \begin{cases} q^{3r^2+r} & \text{if } k = 3r-1, \\ q^{3r^2-r} & \text{if } k = 3r, \\ -q^{3r^2+r} - q^{3r^2+5r+2} & \text{if } k = 3r+1 \end{cases} \quad \text{and} \quad \beta_k = \frac{q^{k^2}}{(q^2)_{2k}} \tag{2.18}$$

and

$$\alpha_k = \begin{cases} q^{3r^2-2r} & \text{if } k = 3r-1, \\ q^{3r^2+2r} & \text{if } k = 3r, \\ -q^{3r^2+4r+1} - q^{3r^2+2r} & \text{if } k = 3r+1 \end{cases} \quad \text{and} \quad \beta_k = \frac{q^{k^2+k}}{(q^2)_{2k}}. \tag{2.19}$$

Next we have two Bailey pairs relative to q from work of Warnaar [34]:

$$\alpha_k = (-1)^{\lfloor \frac{4k+1}{3} \rfloor} q^{\frac{k(2k-1)}{3}} \frac{(1-q^{2k+1})}{(1-q)} \chi(k \not\equiv 1 \pmod 3) \quad \text{and} \quad \beta_k = \frac{1}{(q)_{2k}} \tag{2.20}$$

and

$$\alpha_k = (-1)^{\lfloor \frac{4k+1}{3} \rfloor} q^{\frac{k(k-2)}{3}} \frac{(1-q^{2k+1})}{(1-q)} \chi(k \not\equiv 1 \pmod 3) \quad \text{and} \quad \beta_k = \frac{q^{k^2-k}}{(q)_{2k}}. \tag{2.21}$$

Finally, we have two Bailey pairs relative to q^2. These are [32, Eq. (2.34), $e \to 0$ and $e \to \infty$]:

$$\alpha_k = \begin{cases} 0 & \text{if } k = 3r-1, \\ \dfrac{(1-q^{6r+2})}{1-q^2} q^{3r^2-r} & \text{if } k = 3r, \\ -\dfrac{(1-q^{6r+4})}{1-q^2} q^{3r^2+r} & \text{if } k = 3r+1 \end{cases} \quad \text{and} \quad \beta_k = \frac{q^{k^2}}{(q^2)_{2k}} \tag{2.22}$$

and

$$\alpha_k = \begin{cases} 0 & \text{if } k = 3r-1, \\ \dfrac{(1-q^{6r+2})}{1-q^2} q^{6r^2+r} & \text{if } k = 3r, \\ -\dfrac{(1-q^{6r+4})}{1-q^2} q^{6r^2+5r+1} & \text{if } k = 3r+1 \end{cases} \quad \text{and} \quad \beta_k = \frac{1}{(q^2)_{2k}}. \tag{2.23}$$

To these we add two Bailey pairs relative to q^2 which we were not able to locate in the literature.

Lemma 2.6. *The following two sequences are Bailey pairs relative to q^2:*

$$\alpha_k = \begin{cases} \dfrac{(1-q^{k+1})}{1-q} \left(q^{3r^2-2r} - q^{3r^2-r} \right) & \text{if } k = 3r-1, \\ \dfrac{(1-q^{k+1})}{1-q} q^{3r^2+r} & \text{if } k = 3r, \\ -\dfrac{(1-q^{k+1})}{1-q} q^{3r^2+2r} & \text{if } k = 3r+1 \end{cases} \quad \text{and} \quad \beta_k = \frac{q^{k^2+k}(1+q)}{(q^2)_{2k}(1+q^{k+1})} \tag{2.24}$$

and

$$\alpha_k = \begin{cases} \dfrac{(1-q^{k+1})}{1-q} \left(q^{6r^2-r} - q^{6r^2-2r} \right) & \text{if } k = 3r-1, \\ \dfrac{(1-q^{k+1})}{1-q} q^{6r^2+2r} & \text{if } k = 3r, \\ -\dfrac{(1-q^{k+1})}{1-q} q^{6r^2+7r+2} & \text{if } k = 3r+1 \end{cases} \quad \text{and} \quad \beta_k = \frac{(1+q)}{(q^2)_{2k}(1+q^{k+1})}. \tag{2.25}$$

Proof. We require [25, Eq. (2.4)], which says that if (α_k, β_k) is a Bailey pair relative to a, then $(\alpha_k^\dagger, \beta_k^\dagger)$ is a Bailey pair relative to aq, where

$$\alpha_k^\dagger = \frac{(1-aq^{2k+1})(aq/b)_k(-b)^k q^{\binom{k}{2}}}{(1-aq)(bq)_k} \sum_{j=0}^{k} \frac{(b)_j}{(aq/b)_j} (-b)^{-j} q^{-\binom{j}{2}} \alpha_j \tag{2.26}$$

and

$$\beta_k^\dagger = \frac{(b)_k}{(bq)_k} \beta_k. \tag{2.27}$$

We apply this with $b = -q$ to the Bailey pair relative to q in (2.19). The resulting $\beta_k^\dagger$ is clearly the β_k in (2.24), while

$$\alpha_k^\dagger = \frac{(1-q^{k+1})}{1-q} q^{\binom{k+1}{2}} \sum_{j=0}^{k} q^{-\binom{j+1}{2}} \alpha_j. \tag{2.28}$$

An induction argument on k confirms that $\alpha_k^\dagger$ equals α_k from (2.24).

Next we recall that if (α_k, β_k) is a Bailey pair relative to q^2, then its dual [4] (α_k^d, β_k^d) is also a Bailey pair relative to q^2, where

$$\alpha_k^d = q^{k^2+2k} \alpha_k(q^{-1}) \quad \text{and} \quad \beta_k^d = q^{-k^2-3k} \beta_k(q^{-1}). \tag{2.29}$$

Using the fact that

$$(q^{-2}; q^{-1})_{2k} = q^{-2k^2-3k} (q^2)_{2k}, \tag{2.30}$$

a short calculation shows that (2.25) is the dual of (2.24). □

2.3. **The key results.** Here we state and prove the key results relating (1.6), (1.7), (1.11)–(1.16), (1.24) and (1.25) to finite polynomial sums at roots of unity. We have chosen to present these results following the order in which the Bailey pairs in Section 2.2 were introduced.

Theorem 2.7. *Let (α_k, β_k) denote the Bailey pair in (2.16). Then for $p \geq 1$ and q any primitive Nth root of unity we have*

$$H_p^{(6)}(q) = \sum_{k=0}^{N-1} (-1)^k q^{\binom{k+1}{2} + (p-1)(k^2+k)} \alpha_k. \tag{2.31}$$

Theorem 2.8. *Let (α_k, β_k) denote the Bailey pair in (2.17). Then for $p \geq 1$ and q any primitive Nth root of unity we have*

$$H_p^{(7)}(q) = q \sum_{k=0}^{N-1} (-1)^k q^{\binom{k+1}{2} + (p-1)(k^2+k)} \alpha_k. \tag{2.32}$$

Theorem 2.9. *Let (α_k, β_k) denote the Bailey pair in (2.18). Then for $p \geq 1$ and q any primitive Nth root of unity we have*

$$H_p^{(3)}(q) = q^{-1} \sum_{k=0}^{N-1} (-1)^k q^{\binom{k+1}{2} + (p-1)(k^2+k)} \alpha_k. \tag{2.33}$$

Theorem 2.10. *Let (α_k, β_k) denote the Bailey pair in (2.19). Then for $p \geq 1$ and q any primitive Nth root of unity we have*

$$H_p^{(4)}(q) = \sum_{k=0}^{N-1} (-1)^k q^{\binom{k+1}{2} + (p-1)(k^2+k)} \alpha_k. \tag{2.34}$$

Theorem 2.11. *Let (α_k, β_k) denote the Bailey pair in (2.20). Then for $p \geq 1$ and q any primitive Nth root of unity we have*

$$H_p^{(8)}(q) = -q(1-q) \sum_{k=0}^{N-1} (-1)^k q^{\binom{k+1}{2} + (p-1)(k^2+k)} \alpha_k. \tag{2.35}$$

Theorem 2.12. *Let (α_k, β_k) denote the Bailey pair in (2.21). Then for $p \geq 1$ and q any primitive Nth root of unity we have*

$$H_p^{(1)}(q) = -q^{-1}(1-q)\sum_{k=0}^{N-1}(-1)^k q^{\binom{k+1}{2}+(p-1)(k^2+k)}\alpha_k. \tag{2.36}$$

Theorem 2.13. *Let (α_k, β_k) denote the Bailey pair in (2.22). For $p \geq 1$ and q any primitive Nth root of unity we have*

$$H_p^{(4)}(q) = q^{p-1}\sum_{k=0}^{N-1}(-1)^k q^{-\binom{k+1}{2}}(q^{p(k^2+2k)}\alpha_k)^*. \tag{2.37}$$

Theorem 2.14. *Let (α_k, β_k) denote the Bailey pair in (2.23). For $p \geq 1$ and q any primitive Nth root of unity we have*

$$H_p^{(9)}(q) = q^{p}\sum_{k=0}^{N-1}(-1)^k q^{-\binom{k+1}{2}}(q^{p(k^2+2k)}\alpha_k)^*. \tag{2.38}$$

For the last two results we define

$$\widehat{\alpha}_k = \frac{(1-q)}{1-q^{k+1}}\alpha_k. \tag{2.39}$$

Theorem 2.15. *Let (α_k, β_k) denote the Bailey pair in (2.24). For $p \geq 1$ and q any primitive Nth root of unity we have*

$$-\frac{1}{2} + H_p^{(2)}(q) = q^p\left(\sum_{k=0}^{N-2}(-1)^k q^{-\binom{k+1}{2}+p(k^2+2k)}\widehat{\alpha}_k + \frac{1}{2}q^{-p}\widehat{\alpha}_{N-1}\right). \tag{2.40}$$

Theorem 2.16. *Let (α_k, β_k) denote the Bailey pair in (2.25). For $p \geq 1$ and q any primitive Nth root of unity we have*

$$-\frac{1}{2} + H_p^{(10)}(q) = q^p\left(\sum_{k=0}^{N-2}(-1)^k q^{-\binom{k+1}{2}+p(k^2+2k)}\widehat{\alpha}_k + \frac{1}{2}q^{-p}\widehat{\alpha}_{N-1}\right). \tag{2.41}$$

Proof of Theorem 2.7. Let (α_k, β_k) denote the Bailey pair in (2.16). Using (2.3) we compute that

$$\beta_{N-1-n_p} = \frac{(q^{1-2N})_{2n_p}q^{4Nn_p-2n_p^2-n_p}}{(q^2)_{2N-2}}. \tag{2.42}$$

Inserting this into (2.7) we obtain

$$\begin{aligned}\sum_{N-1\geq n_p\geq\cdots\geq n_1\geq 0}&\frac{q^{(N-n_p-1)(N-n_p)}(q^{1-2N})_{2n_p}q^{4Nn_p-2n_p^2-n_p}}{(q)_{n_p}}\prod_{i=1}^{p-1}q^{(N-n_i-1)(N-n_i)}\begin{bmatrix}n_{i+1}\\ n_i\end{bmatrix}\\ &= \frac{(q^2)_{2N-2}}{(q)_{N-1}(q^2)_{N-1}}\sum_{k=0}^{N-1}\frac{(q^{1-N})_k}{(q^{1+N})_k}(-1)^k q^{Nk-\binom{k+1}{2}+p(k^2+k)}\alpha_k.\end{aligned} \tag{2.43}$$

Now we let q be a primitive Nth root of unity. The terms $\frac{(q^2)_{2N-2}}{(q)_{N-1}(q^2)_{N-1}}$ and $\frac{(q^{1-N})_k}{(q^{1+N})_k}$ become 1, as do all other instances of q^N. The result is (2.31). □

Proofs of Theorems 2.8–2.12. The proofs of Theorems 2.8–2.12 closely resemble the proof of Theorem 2.7. The only difference is the initial Bailey pair (α_k, β_k). Therefore we limit ourselves to noting the expression for β_{N-1-n_p} in each case. For the Bailey pairs in (2.17)–(2.21), we have, respectively,

$$\beta_{N-1-n_p} = \frac{(q^{1-2N})_{2n_p} q^{4Nn_p-2n_p^2-2n_p+N-1}}{(q^2)_{2N-2}}, \tag{2.44}$$

$$\beta_{N-1-n_p} = \frac{(q^{1-2N})_{2n_p} q^{4Nn_p-2n_p^2-n_p+(N-n_p-1)^2}}{(q^2)_{2N-2}}, \tag{2.45}$$

$$\beta_{N-1-n_p} = \frac{(q^{1-2N})_{2n_p} q^{4Nn_p-2n_p^2-n_p+(N-n_p-1)^2+(N-n_p-1)}}{(q^2)_{2N-2}}, \tag{2.46}$$

$$\beta_{N-1-n_p} = \frac{(q^{2-2N})_{2n_p} q^{4Nn_p-2n_p^2-3n_p}}{(q)_{2N-2}} \tag{2.47}$$

and

$$\beta_{N-1-n_p} = \frac{(q^{2-2N})_{2n_p} q^{4Nn_p-2n_p^2-3n_p+(N-1-n_p)^2-(N-1-n_p)}}{(q)_{2N-2}}. \tag{2.48}$$

Using these in (2.3) and calculating as in the proof of Theorem 2.7 we obtain (2.32)–(2.36). □

The proofs of Theorems 2.13 and 2.14 are similar, the only difference being the use of Proposition 2.4 instead of Proposition 2.3.

Proofs of Theorems 2.13–2.14. Let (α_k, β_k) denote the Bailey pair in (2.22). Note that the β_k is the same as in (2.18), and so β_{N-1-n_p} is given by (2.45). Using this in (2.12) we have

$$\sum_{N-1\geq n_p\geq\cdots\geq n_1\geq 0} \frac{q^{(N-n_p)^2-1}(q^{1-2N})_{2n_p} q^{4Nn_p-2n_p^2-n_p+(N-n_p-1)^2}}{(q)_{n_p}} \prod_{i=1}^{p-1} q^{(N-n_i-1)(N-n_i+1)} \begin{bmatrix} n_{i+1} \\ n_i \end{bmatrix}$$
$$= \frac{(q^2)_{2N-2}}{(q)_{N-1}(q^2)_{N-1}} \sum_{k=0}^{N-1} \frac{(q^{1-N})_k}{(q^{1+N})_k} (-1)^k q^{Nk-\binom{k+1}{2}} (q^{p(k^2+2k)}\alpha_k)^*. \tag{2.49}$$

Letting q be a primitive Nth root of unity gives (2.37). Theorem 2.14 follows in the same way using the Bailey pair in (2.23). □

Proofs of Theorems 2.15–2.16. For Theorem 2.15 we use (2.24) in Proposition 2.5 to obtain

$$\sum_{N-1\geq n_p\geq\cdots\geq n_1\geq 0} \frac{q^{(N-n_p-1)(N-n_p+1)} q^{2Nn_p-n_p^2+N^2-N}(q^{1-2N})_{2n_p}}{(1+q^{N-n_p})(q)_{n_p}} \prod_{i=1}^{p-1} q^{(N-n_i-1)(N-n_i+1)} \begin{bmatrix} n_{i+1} \\ n_i \end{bmatrix}$$
$$= \frac{(q)_{2N-1}}{(q)_{N-1}(q)_N} \sum_{k=0}^{N-1} \frac{(q^{1-N})_k}{(q^{2+N})_k} (-1)^k q^{Nk-\binom{k+1}{2}+p(k^2+2k)} \alpha_k. \tag{2.50}$$

Letting q be a primitive Nth root of unity the left-hand side of (2.50) becomes

$$q^{-p} \sum_{n_p\geq\cdots\geq n_1\geq 0} \frac{q^{n_p}(q^{n_p+1})_{n_p}}{(1+q^{n_p})} \prod_{i=1}^{p-1} q^{n_i^2} \begin{bmatrix} n_{i+1} \\ n_i \end{bmatrix}. \tag{2.51}$$

Note that

$$\frac{(q^{n_p+1})_{n_p}}{(1+q^{n_p})} = \begin{cases} (q^{n_p})_{n_p} & \text{if } n_p > 0 \\ \dfrac{1}{2} & \text{if } n_p = 0 \end{cases} \tag{2.52}$$

and so (2.51) when q is a primitive Nth root of unity is

$$q^{-p}\left(-\frac{1}{2} + \sum_{n_p \geq \cdots \geq n_1 \geq 0} q^{n_p}(q^{n_p})_{n_p} \prod_{i=1}^{p-1} q^{n_i^2} \begin{bmatrix} n_{i+1} \\ n_i \end{bmatrix}\right). \tag{2.53}$$

Now letting q be a primitive Nth root of unity on the right-hand side of (2.50), the prefactor becomes 1 as usual. As for the term $\frac{(q^{1-N})_k}{(q^{2+N})_k}$, it does not become 1 as before. But when multiplied by $(1-q^{k+1})/(1-q)$, it becomes 1 for $k < N-1$ while for $k = N-1$ it is $\frac{(1-q^N)}{(1-q^{2N})} \to \frac{1}{2}$. This observation accounts for the use of $\widehat{\alpha}_k$ and the splitting of the sum on the right-hand side of (2.40). Here, we also use that $q^{-\binom{N}{2}+pN^2} = (-1)^{N+1}$. This gives Theorem 2.15. Theorem 2.16 follows in the same way using (2.25) in Proposition 2.5. □

3. Incomplete quadratic Gauss sums

3.1. **Properties for incomplete quadratic Gauss sums.** We begin with a general result on an incomplete quadratic Gauss sum. For simplicity, we write $\chi = \chi_{\boldsymbol{p}}^{\boldsymbol{\ell}}$. Recall that $P := p_1p_2p_3$.

Lemma 3.1. *Assume that*

(i) $P \equiv 2 \pmod 4$,

(ii) if $\chi(n) \neq 0$, *then* n *is odd*

(iii) χ *is even modulo* P, *i.e.,* $\chi(P-n) = \chi(n)$.

If q *is a primitive* N*th root of unity, then*

$$2\sum_{n=0}^{PN}\left(1-\frac{n}{PN}\right)\chi(n)q^{\frac{n^2}{4P}} = \sum_{n=0}^{PN}\chi(n)q^{\frac{n^2}{4P}}.$$

Proof. We have

$$\begin{aligned}
&2\sum_{n=0}^{PN}\left(1-\frac{n}{PN}\right)\chi(n)q^{\frac{n^2}{4P}} \\
&= \sum_{n=0}^{PN}\left(1-\frac{n}{PN}\right)\chi(n)q^{\frac{n^2}{4P}} + \sum_{n=0}^{PN}\left(1-\frac{n}{PN}\right)\chi(n)q^{\frac{n^2}{4P}} \\
&= \sum_{n=0}^{PN}\left(1-\frac{n}{PN}\right)\chi(n)q^{\frac{n^2}{4P}} + \sum_{n=0}^{PN}\left(1-\frac{(PN-n)}{PN}\right)\chi(PN-n)q^{\frac{(PN-n)^2}{4P}} \\
&= \sum_{n=0}^{PN}\left(1-\frac{n}{PN}\right)\chi(n)q^{\frac{n^2}{4P}} + \sum_{n=0}^{PN}\frac{n}{PN}\chi(PN-n)q^{\frac{(P^2N^2-2nPN+n^2)}{4P}}
\end{aligned}$$

$$= \sum_{n=0}^{PN} \left(1 - \frac{n}{PN}\right) \chi(n) q^{\frac{n^2}{4P}} + \sum_{n=0}^{PN} \frac{n}{PN} q^{\frac{n^2}{4P}} \chi(PN-n) q^{\frac{N(PN-2n)}{4}}. \tag{3.1}$$

Consider the term

$$\chi(PN-n) q^{\frac{N(PN-2n)}{4}}.$$

We have two cases according to the parity of N. First, if N is odd then since $P \equiv 2 \pmod 4$ and n is odd we have that $PN - 2n$ is a multiple of 4. Therefore $q^{\frac{N(PN-2n)}{4}} = 1$. Moreover, by (iii) we have $\chi(PN-n) = \chi(n)$. Hence

$$\chi(PN-n) q^{\frac{N(PN-2n)}{4}} = \chi(n).$$

If N is even, then $PN - 2n \equiv 2 \pmod 4$ and so $q^{\frac{N(PN-2n)}{4}} = -1$. Now since N is even we have $\chi(PN-n) = -\chi(n)$. Once again, we find that

$$\chi(PN-n) q^{\frac{N(PN-2n)}{4}} = \chi(n).$$

Using this in (3.1) we have

$$\begin{aligned} 2\sum_{n=0}^{PN} \left(1 - \frac{n}{PN}\right) \chi(n) q^{\frac{n^2}{4P}} &= \sum_{n=0}^{PN} \left(1 - \frac{n}{PN}\right) \chi(n) q^{\frac{n^2}{4P}} + \sum_{n=0}^{PN} \frac{n}{PN} q^{\frac{n^2}{4P}} \chi(n) \\ &= \sum_{n=0}^{PN} \chi(n) q^{\frac{n^2}{4P}}, \end{aligned}$$

as desired. □

Remark 3.2. Note that (i) and (ii) in Lemma 3.1 are satisfied for all of the functions $\chi_{\boldsymbol{p}}^{\boldsymbol{\ell}}$ considered in Theorem 1.2. Moreover, $p_1 = 2$ and so by [18, (3.8)] the involution σ_1 fixes $\boldsymbol{\ell} = (1, 1, \ell_3)$, i.e., $\chi_{\boldsymbol{p}}^{\boldsymbol{\ell}}(n+P) = -\chi_{\boldsymbol{p}}^{\boldsymbol{\ell}}(n)$. Now replace n with $-n$ and use the fact that $\chi_{\boldsymbol{p}}^{\boldsymbol{\ell}}$ is odd to obtain (iii) in Lemma 3.1. Finally, note that $\chi_{\boldsymbol{p}}^{\boldsymbol{\ell}}$ is odd modulo NP for all even N and even modulo NP for all odd N.

Next, we recall a result (see [16, Proposition 3] or [29, Corollary 3.9]) which explicitly computes the limiting values of the Eichler integrals $\widetilde{\Phi}_{\boldsymbol{p}}^{\boldsymbol{\ell}}(\tau)$ as $\tau \to \frac{M}{N}$.

Proposition 3.3. *Let M and N be coprime positive integers. We have*

$$\lim_{\tau \to \frac{M}{N}} \widetilde{\Phi}_{\boldsymbol{p}}^{\boldsymbol{\ell}}(\tau) = \sum_{n=0}^{PN} \chi_{\boldsymbol{p}}^{\boldsymbol{\ell}} \left(1 - \frac{n}{PN}\right) \zeta_N^{M\frac{n^2}{4P}}.$$

3.2. **Relations between incomplete quadratic Gauss sums.** For the fixed tuples $\boldsymbol{p} = (p_1, p_2, p_3)$ of pairwise coprime positive integers and $\boldsymbol{\ell} = (\ell_1, \ell_2, \ell_3) \in \mathbb{Z}^3$ satisfying $0 < \ell_j < p_j$, we define $m(\boldsymbol{\epsilon}) \in \{0, \ldots, 2P-1\}$ such that

$$m(\boldsymbol{\epsilon}) \equiv P\left(1 + \sum_{j=1}^{3} \frac{\epsilon_j \ell_j}{p_j}\right) \pmod{2P}. \tag{3.2}$$

Note that $m(\boldsymbol{\epsilon})^2 \pmod{2P}$ is independent of $\boldsymbol{\epsilon} \in \{\pm 1\}^3$. We write

$$T = T(q) = -\sum_{\boldsymbol{\epsilon}\in\{\pm1\}^3} \epsilon_1\epsilon_2\epsilon_3\, T_{\boldsymbol{\epsilon}}, \qquad S = S(q) = -\sum_{\boldsymbol{\epsilon}\in\{\pm1\}^3} \epsilon_1\epsilon_2\epsilon_3\, S_{\boldsymbol{\epsilon}}$$

where

$$T_{\boldsymbol{\epsilon}} = T_{\boldsymbol{\epsilon}}(q) := \sum_{\substack{n\equiv m(\boldsymbol{\epsilon}) \bmod 2P\\ 0\le n\le 2p_3N}} q^{\frac{n^2}{4P}}, \qquad S_{\boldsymbol{\epsilon}} = S_{\boldsymbol{\epsilon}}(q) := \sum_{\substack{n\equiv m(\boldsymbol{\epsilon}) \bmod 2P\\ 2p_3N< n\le 4p_3N}} q^{\frac{n^2}{4P}}.$$

If we write $n = k2P + m(\boldsymbol{\epsilon})$ for some positive integer k, then

$$T_{\boldsymbol{\epsilon}} = \sum_{k=0}^{\left\lfloor \frac{2p_3N-m(\boldsymbol{\epsilon})}{2P}\right\rfloor} q^{Pk^2+m(\boldsymbol{\epsilon})k+\frac{m(\boldsymbol{\epsilon})^2}{4P}}, \qquad S_{\boldsymbol{\epsilon}} = \sum_{k=\left\lceil \frac{2p_3N-m(\boldsymbol{\epsilon})}{2P}\right\rceil}^{\left\lfloor \frac{4p_3N-m(\boldsymbol{\epsilon})}{2P}\right\rfloor} q^{Pk^2+m(\boldsymbol{\epsilon})k+\frac{m(\boldsymbol{\epsilon})^2}{4P}} \tag{3.3}$$

and so $T_{\boldsymbol{\epsilon}}$, $S_{\boldsymbol{\epsilon}} \in q^{\frac{m(\boldsymbol{\epsilon})^2}{4P}}\mathbb{Z}[q]$. The sums (3.3) satisfy the following relations when evaluated at a root of unity.

Proposition 3.4. *Let $\boldsymbol{p} = (2, 3, p_3)$ with $(6, p_3) = 1$, $\boldsymbol{\ell} = (1, 1, \ell_3) \in \mathbb{Z}^3$ satisfy $0 < \ell_3 < p_3$ and $\boldsymbol{\epsilon} := (\epsilon_1, \epsilon_2, \epsilon_3)$, $\boldsymbol{\epsilon}' := (\epsilon_1', \epsilon_2', \epsilon_3') \in \{\pm1\}^3$. If M and N are coprime positive integers and $q = e^{\frac{2\pi i M}{N}}$, then the following are true:*

(i) If $\frac{\epsilon_1+\epsilon_1'}{2} \equiv N \pmod 2$, $\epsilon_2+\epsilon_2' \equiv N \pmod 3$ and $\epsilon_3' = -\epsilon_3$, then $\frac{m(\boldsymbol{\epsilon})+m(\boldsymbol{\epsilon}')}{2P} \equiv \frac{N}{6} \pmod 1$ and

$$T_{\boldsymbol{\epsilon}} = e^{\frac{\pi i (p_3N-m(\boldsymbol{\epsilon}'))M}{3}} T_{\boldsymbol{\epsilon}'}.$$

(ii) If $\frac{\epsilon_1-\epsilon_1'}{2} \equiv N \pmod 2$, $\epsilon_2-\epsilon_2' \equiv N \pmod 3$ and $\epsilon_3' = \epsilon_3$, then $\frac{m(\boldsymbol{\epsilon})-m(\boldsymbol{\epsilon}')}{2P} \equiv \frac{N}{6} \pmod 1$ and

$$S_{\boldsymbol{\epsilon}} = e^{\frac{\pi i (p_3N+m(\boldsymbol{\epsilon}'))M}{3}} T_{\boldsymbol{\epsilon}'}.$$

(iii) If $\frac{\epsilon_1+\epsilon_1'}{2} \equiv 0 \pmod 2$, $\epsilon_2+\epsilon_2' \equiv 2N \pmod 3$ and $\epsilon_3 = -\epsilon_3'$, then $\frac{m(\boldsymbol{\epsilon})+m(\boldsymbol{\epsilon}')}{2P} \equiv \frac{N}{3} \pmod 1$ and

$$S_{\boldsymbol{\epsilon}} = e^{\frac{2\pi i (2p_3N-m(\boldsymbol{\epsilon}'))M}{3}} T_{\boldsymbol{\epsilon}'}.$$

(iv) If $\frac{\epsilon_1+\epsilon_1'}{2} \equiv N \pmod 2$, $\epsilon_2+\epsilon_2' \equiv 0 \pmod 3$ and $\epsilon_3 = -\epsilon_3'$, then $\frac{m(\boldsymbol{\epsilon})+m(\boldsymbol{\epsilon}')}{2P} \equiv \frac{N}{2} \pmod 1$ and

$$S_{\boldsymbol{\epsilon}} = e^{\pi i (p_3N-m(\boldsymbol{\epsilon}'))M} S_{\boldsymbol{\epsilon}'}.$$

Proof. We begin with (i). By (3.2) and simplifying, we have

$$m(\boldsymbol{\epsilon}) + m(\boldsymbol{\epsilon}') \equiv 3p_3(\epsilon_1+\epsilon_1') + 2p_3(\epsilon_2+\epsilon_2') + 6\ell_3(\epsilon_3+\epsilon_3') \pmod{2P}. \tag{3.4}$$

The conditions on $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$ combined with (3.4) now imply that

$$\frac{m(\boldsymbol{\epsilon})+m(\boldsymbol{\epsilon}')}{2P} \equiv \frac{N}{6} \pmod 1. \tag{3.5}$$

Since $m(\boldsymbol{\epsilon})$, $m(\boldsymbol{\epsilon}') \in [0, 2P)$, it follows that $\frac{m(\boldsymbol{\epsilon})}{2P}$, $\frac{m(\boldsymbol{\epsilon}')}{2P} \in [0, 1)$. By (3.5), we have

$$\frac{N}{6} - \frac{m(\boldsymbol{\epsilon})}{2P} - \left\lfloor \frac{N}{6} - \frac{m(\boldsymbol{\epsilon})}{2P} \right\rfloor = \frac{m(\boldsymbol{\epsilon}')}{2P} \tag{3.6}$$

and so rearranging terms and applying the floor function yields

$$\left\lfloor \frac{N}{6} - \frac{m(\boldsymbol{\epsilon})}{2P} \right\rfloor = \left\lfloor \frac{N}{6} - \frac{m(\boldsymbol{\epsilon}')}{2P} \right\rfloor. \tag{3.7}$$

From (3.3), we compute

$$\begin{aligned} T_{\boldsymbol{\epsilon}} &= \sum_{k=0}^{\left\lfloor \frac{N}{6} - \frac{m(\boldsymbol{\epsilon})}{2P} \right\rfloor} q^{Pk^2 + m(\boldsymbol{\epsilon})k + \frac{m(\boldsymbol{\epsilon})^2}{4P}} \\ &= \sum_{k=0}^{\left\lfloor \frac{N}{6} - \frac{m(\boldsymbol{\epsilon})}{2P} \right\rfloor} q^{P\left(\left\lfloor \frac{N}{6} - \frac{m(\boldsymbol{\epsilon})}{2P} \right\rfloor - k\right)^2 + m(\boldsymbol{\epsilon})\left(\left\lfloor \frac{N}{6} - \frac{m(\boldsymbol{\epsilon})}{2P} \right\rfloor - k\right) + \frac{m(\boldsymbol{\epsilon})^2}{4P}} \\ &= q^{P\left\lfloor \frac{N}{6} - \frac{m(\boldsymbol{\epsilon})}{2P} \right\rfloor^2 + m(\boldsymbol{\epsilon})\left\lfloor \frac{N}{6} - \frac{m(\boldsymbol{\epsilon})}{2P} \right\rfloor + \frac{m(\boldsymbol{\epsilon})^2 - m(\boldsymbol{\epsilon}')^2}{4P}} \\ &\qquad \times \sum_{k=0}^{\left\lfloor \frac{N}{6} - \frac{m(\boldsymbol{\epsilon})}{2P} \right\rfloor} q^{k^2 - \left(2P\left\lfloor \frac{N}{6} - \frac{m(\boldsymbol{\epsilon})}{2P} \right\rfloor + m(\boldsymbol{\epsilon})\right)k + \frac{m(\boldsymbol{\epsilon}')^2}{4P}} \\ &= q^{\frac{N(p_3 N - m(\boldsymbol{\epsilon}'))}{6}} \sum_{k=0}^{\left\lfloor \frac{N}{6} - \frac{m(\boldsymbol{\epsilon}')}{2P} \right\rfloor} q^{k^2 + m(\boldsymbol{\epsilon}')k + \frac{PNk}{3} + \frac{m(\boldsymbol{\epsilon}')^2}{4P}} \end{aligned} \tag{3.8}$$

where we have applied the substitution $k \mapsto \left\lfloor \frac{N}{6} - \frac{m(\boldsymbol{\epsilon})}{2P} \right\rfloor - k$ and used (3.6) and (3.7). Letting $q = e^{\frac{2\pi i M}{N}}$ in (3.8) and simplifying yields the result.

For (ii), a computation similar to that of (3.4) and the conditions on $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$ imply

$$\frac{m(\boldsymbol{\epsilon}) - m(\boldsymbol{\epsilon}')}{2P} \equiv \frac{N}{6} \pmod 1. \tag{3.9}$$

By (3.9), we have

$$\frac{N}{6} - \frac{m(\boldsymbol{\epsilon})}{2P} - \left\lfloor \frac{N}{6} - \frac{m(\boldsymbol{\epsilon})}{2P} \right\rfloor = -\frac{m(\boldsymbol{\epsilon}')}{2P} + 1 \tag{3.10}$$

and so again rearranging terms and applying the floor function leads to

$$\left\lfloor \frac{N}{3} - \frac{m(\boldsymbol{\epsilon})}{2P} \right\rfloor - \left\lfloor \frac{N}{6} - \frac{m(\boldsymbol{\epsilon})}{2P} \right\rfloor = \left\lfloor \frac{N}{6} - \frac{m(\boldsymbol{\epsilon}')}{2P} \right\rfloor + 1. \tag{3.11}$$

From (3.3), we compute

$$\begin{aligned}
S_\epsilon &= \sum_{k=\left\lfloor \frac{N}{6}-\frac{m(\boldsymbol{\epsilon})}{2P}\right\rfloor+1}^{\left\lfloor \frac{N}{3}-\frac{m(\boldsymbol{\epsilon})}{2P}\right\rfloor} q^{Pk^2+m(\boldsymbol{\epsilon})k+\frac{m(\boldsymbol{\epsilon})^2}{4P}} \\
&= \sum_{k=0}^{\left\lfloor \frac{N}{3}-\frac{m(\boldsymbol{\epsilon})}{2P}\right\rfloor-\left\lfloor \frac{N}{6}-\frac{m(\boldsymbol{\epsilon})}{2P}\right\rfloor-1} q^{P\left(\left\lfloor \frac{N}{6}-\frac{m(\boldsymbol{\epsilon})}{2P}\right\rfloor+1+k\right)^2+m(\boldsymbol{\epsilon})\left(\left\lfloor \frac{N}{6}-\frac{m(\boldsymbol{\epsilon})}{2P}\right\rfloor+1+k\right)+\frac{m(\boldsymbol{\epsilon})^2}{4P}} \\
&= q^{P\left(\left\lfloor \frac{N}{6}-\frac{m(\boldsymbol{\epsilon})}{2P}\right\rfloor+1\right)^2+m(\boldsymbol{\epsilon})\left(\left\lfloor \frac{N}{6}-\frac{m(\boldsymbol{\epsilon})}{2P}\right\rfloor+1\right)+\frac{m(\boldsymbol{\epsilon})^2}{4P}} \\
&\qquad \times \sum_{k=0}^{\left\lfloor \frac{N}{3}-\frac{m(\boldsymbol{\epsilon})}{2P}\right\rfloor-\left\lfloor \frac{N}{6}-\frac{m(\boldsymbol{\epsilon})}{2P}\right\rfloor-1} q^{Pk^2+\left(m(\boldsymbol{\epsilon})+2P\left\lfloor \frac{N}{6}-\frac{m(\boldsymbol{\epsilon})}{2P}\right\rfloor+2P\right)k+\frac{m(\boldsymbol{\epsilon}')^2}{4P}} \\
&= q^{\frac{N(p_3N+m(\boldsymbol{\epsilon}'))}{6}} \sum_{k=0}^{\left\lfloor \frac{N}{6}-\frac{m(\boldsymbol{\epsilon}')}{2P}\right\rfloor} q^{Pk^2+m(\boldsymbol{\epsilon}')k+\frac{PNk}{3}+\frac{m(\boldsymbol{\epsilon}')^2}{4P}}
\end{aligned} \tag{3.12}$$

where we have applied the substitution $k \mapsto \left\lfloor \frac{N}{6} - \frac{m(\boldsymbol{\epsilon})}{2P} \right\rfloor + 1 + k$ and used (3.10) and (3.11). Letting $q = e^{\frac{2\pi i M}{N}}$ in (3.12) and simplifying yields the result.

For (iii), a computation similar to that of (3.4) and the conditions on $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$ imply

$$\frac{m(\boldsymbol{\epsilon}) + m(\boldsymbol{\epsilon}')}{2P} \equiv \frac{N}{3} \pmod 1. \tag{3.13}$$

By (3.13), we have

$$\frac{N}{3} - \frac{m(\boldsymbol{\epsilon})}{2P} - \left\lfloor \frac{N}{3} - \frac{m(\boldsymbol{\epsilon})}{2P} \right\rfloor = \frac{m(\boldsymbol{\epsilon}')}{2P} \tag{3.14}$$

and so again rearranging terms and applying the floor function leads to

$$\left\lfloor \frac{N}{3} - \frac{m(\epsilon)}{2P} \right\rfloor - \left\lfloor \frac{N}{6} - \frac{m(\epsilon)}{2P} \right\rfloor - 1 = \left\lfloor \frac{N}{6} - \frac{m(\epsilon')}{2P} \right\rfloor. \tag{3.15}$$

Here, we have used $\lfloor -x\rfloor = -\lfloor x\rfloor - 1$ for $x \in \mathbb{R}\setminus\mathbb{Z}$. From (3.3), we compute

$$\begin{aligned}
S_{\boldsymbol{\epsilon}} &= \sum_{k=\left\lfloor \frac{N}{6}-\frac{m(\boldsymbol{\epsilon})}{2P}\right\rfloor+1}^{\left\lfloor \frac{N}{3}-\frac{m(\boldsymbol{\epsilon})}{2P}\right\rfloor} q^{Pk^2+m(\boldsymbol{\epsilon})k+\frac{m(\boldsymbol{\epsilon})^2}{4P}} \\
&= \sum_{k=0}^{\left\lfloor \frac{N}{3}-\frac{m(\boldsymbol{\epsilon})}{2P}\right\rfloor-\left\lfloor \frac{N}{6}-\frac{m(\boldsymbol{\epsilon})}{2P}\right\rfloor-1} q^{P\left(\left\lfloor \frac{N}{3}-\frac{m(\boldsymbol{\epsilon})}{2P}\right\rfloor-k\right)^2+m(\boldsymbol{\epsilon})\left(\left\lfloor \frac{N}{3}-\frac{m(\boldsymbol{\epsilon})}{2P}\right\rfloor-k\right)+\frac{m(\boldsymbol{\epsilon})^2}{4P}} \\
&= q^{P\left(\left\lfloor \frac{N}{3}-\frac{m(\boldsymbol{\epsilon})}{2P}\right\rfloor\right)^2+m(\boldsymbol{\epsilon})\left\lfloor \frac{N}{3}-\frac{m(\boldsymbol{\epsilon})}{2P}\right\rfloor+\frac{m(\boldsymbol{\epsilon})^2}{4P}} \\
&\quad\times \sum_{k=0}^{\left\lfloor \frac{N}{3}-\frac{m(\boldsymbol{\epsilon})}{2P}\right\rfloor-\left\lfloor \frac{N}{6}-\frac{m(\boldsymbol{\epsilon})}{2P}\right\rfloor-1} q^{Pk^2-\left(m(\boldsymbol{\epsilon})+2P\left\lfloor \frac{N}{3}-\frac{m(\boldsymbol{\epsilon})}{2P}\right\rfloor\right)k+\frac{m(\boldsymbol{\epsilon}')^2}{4P}} \\
&= q^{\frac{N\left(PN/3-m(\boldsymbol{\epsilon}')\right)}{3}} \sum_{k=0}^{\left\lfloor \frac{N}{6}-\frac{m(\boldsymbol{\epsilon}')}{2P}\right\rfloor} q^{Pk^2+m(\boldsymbol{\epsilon}')k+\frac{2PNk}{3}+\frac{m(\boldsymbol{\epsilon}')^2}{4P}}
\end{aligned} \tag{3.16}$$

where we have applied the substitution $k \mapsto \left\lfloor \frac{N}{3} - \frac{m(\boldsymbol{\epsilon})}{2P}\right\rfloor - k$ and used (3.14) and (3.15). Letting $q = e^{\frac{2\pi i M}{N}}$ in (3.16) and simplifying yields the result.

In order to prove (iv), we will use (ii) and (iii). A computation similar to that of (3.4) and the conditions on $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$ imply

$$\frac{m(\boldsymbol{\epsilon})+m(\boldsymbol{\epsilon}')}{2P} \equiv \frac{N}{2} \pmod 1. \tag{3.17}$$

We now distinguish between two different cases. First, if $2N-\epsilon_2 \not\equiv 0 \pmod 3$, we define $\boldsymbol{\epsilon}'' = (\epsilon_1'', \epsilon_2'', \epsilon_3'') \in \{\pm 1\}^3$ by

$$\frac{\epsilon_1+\epsilon_1''}{2} \equiv 0 \pmod 2, \qquad \epsilon_2+\epsilon_2'' \equiv 2N \pmod 3, \qquad \epsilon_3 = -\epsilon_3''. \tag{3.18}$$

Then (iii) with conditions (3.18) and (3.17) imply

$$S_{\boldsymbol{\epsilon}} = e^{\frac{2\pi i\left(2p_3N-m(\boldsymbol{\epsilon}'')\right)M}{3}} T_{\boldsymbol{\epsilon}''} = e^{\frac{2\pi i\left(-2p_3N-m(\boldsymbol{\epsilon}')\right)M}{3}} T_{\boldsymbol{\epsilon}''}. \tag{3.19}$$

Subtracting the equations in (3.18) from the initial assumptions on $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$ yields

$$\frac{\epsilon_1'-\epsilon_1''}{2} \equiv N \pmod 2, \qquad \epsilon_2'-\epsilon_2'' \equiv N \pmod 3, \qquad \epsilon_3' = \epsilon_3''. \tag{3.20}$$

Thus (ii) with conditions (3.20) imply

$$T_{\boldsymbol{\epsilon}''} = e^{\frac{-\pi i\left(p_3N+m(\boldsymbol{\epsilon}'')\right)M}{3}} S_{\boldsymbol{\epsilon}'} = e^{\frac{-\pi i\left(-p_3N+m(\boldsymbol{\epsilon}')\right)M}{3}} S_{\boldsymbol{\epsilon}'}. \tag{3.21}$$

We now combine (3.19) and (3.21) and simplify to obtain the result. Second, if $2N-\epsilon_2 \equiv 0 \bmod 3$, then $N-\epsilon_2 \not\equiv 0 \bmod 3$ and we define $\boldsymbol{\epsilon}''' = (\epsilon_1''', \epsilon_2''', \epsilon_3''') \in \{\pm 1\}^3$ by

$$\frac{\epsilon_1-\epsilon_1'''}{2} \equiv N \pmod 2, \qquad \epsilon_2-\epsilon_2''' \equiv N \pmod 3, \qquad \epsilon_3 = \epsilon_3'''. \tag{3.22}$$

Thus (ii) with conditions (3.22) imply

$$S_{\boldsymbol{\epsilon}} = e^{\frac{\pi i\left(p_3 N + m(\boldsymbol{\epsilon}''')\right)M}{3}} T_{\boldsymbol{\epsilon}'''} = e^{\frac{\pi i\left(-p_3 N - m(\boldsymbol{\epsilon}')\right)M}{3}} T_{\boldsymbol{\epsilon}'''}. \tag{3.23}$$

Subtracting the equations in (3.22) from the initial assumptions on $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$, we obtain

$$\frac{\epsilon_1' + \epsilon_1'''}{2} \equiv 0 \pmod 2, \qquad \epsilon_2' + \epsilon_2''' \equiv 2N \pmod 3, \qquad \epsilon_3' = -\epsilon_3'''. \tag{3.24}$$

Thus (iii) with conditions (3.24) imply

$$T_{\boldsymbol{\epsilon}'''} = e^{\frac{-2\pi i\left(2p_3 N - m(\boldsymbol{\epsilon}''')\right)M}{3}} S_{\boldsymbol{\epsilon}'} = e^{\frac{-2\pi i\left(4p_3 N + m(\boldsymbol{\epsilon}')\right)M}{3}} S_{\boldsymbol{\epsilon}'}. \tag{3.25}$$

We now combine (3.23) and (3.25) and simplify to obtain the result. □

Applying the identities from Proposition 3.4 immediately yields the following result. Henceforth, we use the convention $+$ for $+1$ and $-$ for -1 in the indices of $T_{\boldsymbol{\epsilon}}$ and $S_{\boldsymbol{\epsilon}}$.

Corollary 3.5. *Let $\boldsymbol{p} = (2, 3, p_3)$ with $(6, p_3) = 1$ and $\boldsymbol{\ell} = (1, 1, \ell_3) \in \mathbb{Z}^3$ satisfy $0 < \ell_3 < p_3$. If M and N are coprime positive integers and $q = e^{\frac{2\pi i M}{N}}$, then the following are true:*

(1) If $N \equiv 0 \bmod 6$, then

$$\begin{aligned}
T_{+++} &= e^{-\frac{\pi i p_3 M}{3}} T_{---} = e^{\frac{\pi i p_3 M}{3}} S_{+++} = e^{-\frac{2\pi i p_3 M}{3}} S_{---},\\
T_{++-} &= e^{-\frac{\pi i p_3 M}{3}} T_{--+} = e^{-\frac{2\pi i p_3 M}{3}} S_{++-} = e^{\frac{\pi i p_3 M}{3}} S_{--+},\\
T_{+-+} &= e^{\frac{\pi i p_3 M}{3}} T_{-+-} = e^{-\frac{\pi i p_3 M}{3}} S_{+-+} = e^{\frac{2\pi i p_3 M}{3}} S_{-+-},\\
T_{+--} &= e^{\frac{\pi i p_3 M}{3}} T_{-++} = e^{-\frac{\pi i p_3 M}{3}} S_{+--} = e^{\frac{2\pi i p_3 M}{3}} S_{-++}.
\end{aligned}$$

(2) If $N \equiv 1 \bmod 6$, then

$$\begin{aligned}
T_{+-+} &= T_{+--}, & T_{--+} &= T_{---},\\
T_{-++} &= S_{+-+} = S_{++-}, & T_{-+-} &= S_{+--} = S_{+++},\\
T_{+++} &= S_{--+} = S_{-+-}, & T_{++-} &= S_{---} = S_{-++}.
\end{aligned}$$

(3) If $N \equiv 2 \bmod 6$, then

$$\begin{aligned}
T_{+++} &= -T_{-+-}, & T_{++-} &= -T_{-++},\\
T_{+-+} &= -S_{+++} = S_{---}, & T_{+--} &= -S_{++-} = S_{--+},\\
T_{--+} &= -S_{-++} = S_{+--}, & T_{---} &= -S_{-+-} = S_{+-+}.
\end{aligned}$$

(4) If $N \equiv 3 \bmod 6$, then

$$\begin{aligned}
T_{+++} &= e^{\frac{2\pi M}{3}} T_{+--} = e^{-\frac{2\pi i M}{3}} S_{-++} = e^{-\frac{2\pi i M}{3}} S_{---},\\
T_{++-} &= e^{\frac{2\pi i M}{3}} T_{+-+} = e^{-\frac{2\pi i M}{3}} S_{-+-} = e^{-\frac{2\pi i M}{3}} S_{--+},\\
T_{-++} &= e^{\frac{2\pi i M}{3}} T_{---} = e^{-\frac{2\pi i M}{3}} S_{+++} = e^{-\frac{2\pi i M}{3}} S_{+--},\\
T_{-+-} &= e^{\frac{2\pi i M}{3}} T_{--+} = e^{-\frac{2\pi i M}{3}} - S_{++-} = e^{-\frac{2\pi i M}{3}} - S_{+-+}.
\end{aligned}$$

(5) If $N \equiv 4 \bmod 6$*, then*

$$\begin{aligned} T_{+-+} &= -T_{---}, & T_{+--} &= -T_{--+}, \\ T_{+++} &= -S_{+-+} = S_{-+-}, & T_{++-} &= -S_{+--} = S_{-++}, \\ T_{-++} &= -S_{--+} = S_{++-}, & T_{-+-} &= -S_{---} = S_{+++}. \end{aligned}$$

(6) If $N \equiv 5 \bmod 6$*, then*

$$\begin{aligned} T_{+++} &= T_{++-}, & T_{-++} &= T_{-+-}, \\ T_{--+} &= S_{+++} = S_{+--}, & T_{---} &= S_{++-} = S_{+-+}, \\ T_{+-+} &= S_{-++} = S_{---}, & T_{+--} &= S_{-+-} = S_{--+}. \end{aligned}$$

We now illustrate the previous result.

Corollary 3.6. *Let* $\boldsymbol{p} = (2, 3, p_3)$ *with* $(6, p_3) = 1$ *and* $\boldsymbol{\ell} = (1, 1, \ell_3) \in \mathbb{Z}^3$ *satisfy* $0 < \ell_3 < p_3$ *and write* $\chi = \chi^{(1,1,\ell_3)}_{(2,3,p_3)}$*. If* M *and* N *are coprime positive integers and* $q = e^{\frac{2\pi i M}{N}}$*, then*

$$\sum_{n=0}^{6p_3N} \chi(n) q^{\frac{n^2}{24p_3}} = 4 \sum_{n=0}^{2p_3N} \chi(n) q^{\frac{n^2}{24p_3}}. \tag{3.26}$$

Proof. By Remark 3.2, we obtain

$$\begin{aligned} \sum_{n=0}^{6p_3N} \chi(n) q^{\frac{n^2}{24p_3}} &= -2 \sum_{\boldsymbol{\epsilon} \in \{\pm 1\}^3} \epsilon_1\epsilon_2\epsilon_3 T_{\boldsymbol{\epsilon}} - \sum_{\boldsymbol{\epsilon} \in \{\pm 1\}^3} \epsilon_1\epsilon_2\epsilon_3 S_{\boldsymbol{\epsilon}} \\ &= 2\left(-T_{+++} + T_{++-} + T_{+-+} - T_{+--} + T_{-++} - T_{-+-} - T_{--+} + T_{---}\right) \\ &\quad - S_{+++} + S_{++-} + S_{+-+} - S_{+--} + S_{-++} - S_{-+-} - S_{--+} + S_{---} \end{aligned}$$

and

$$\begin{aligned} \sum_{n=0}^{2p_3N} \chi(n) q^{\frac{n^2}{24p_3}} &= -\sum_{\boldsymbol{\epsilon} \in \{\pm 1\}^3} \epsilon_1\epsilon_2\epsilon_3 T_{\boldsymbol{\epsilon}} \\ &= -T_{+++} + T_{++-} + T_{+-+} - T_{+--} + T_{-++} - T_{-+-} - T_{--+} + T_{---}. \end{aligned}$$

Hence, to verify (3.26) it suffices to prove that

$$\begin{aligned} 0 &= -2\left(-T_{+++} + T_{++-} + T_{+-+} - T_{+--} + T_{-++} - T_{-+-} - T_{--+} + T_{---}\right) \\ &\quad - S_{+++} + S_{++-} + S_{+-+} - S_{+--} + S_{-++} - S_{-+-} - S_{--+} + S_{---}. \end{aligned} \tag{3.27}$$

If $N \not\equiv 0 \pmod 3$, this follows directly from Corollary 3.5 (2), (3), (5) and (6). Otherwise, the terms on the right-hand side of (3.27) have to be rearranged. For example, if $N \equiv 0 \pmod 6$, by Corollary 3.5 (1), the right-hand side of (3.27) is given by

$$\begin{aligned} &T_{+++}\left(2 - 2e^{\frac{\pi i p_3 M}{3}} - e^{-\frac{\pi i p_3 M}{3}} + e^{\frac{2\pi i p_3 M}{3}}\right) + T_{++-}\left(-2 + 2e^{\frac{\pi i p_3 M}{3}} + e^{-\frac{\pi i p_3 M}{3}} - e^{\frac{2\pi i p_3 M}{3}}\right) \\ &+ T_{+-+}\left(-2 + 2e^{-\frac{\pi i p_3 M}{3}} + e^{\frac{\pi i p_3 M}{3}} - e^{-\frac{2\pi i p_3 M}{3}}\right) + T_{+--}\left(2 - 2e^{-\frac{\pi i p_3 M}{3}} - e^{\frac{\pi i p_3 M}{3}} + e^{-\frac{2\pi i p_3 M}{3}}\right) \end{aligned} \tag{3.28}$$

where $M \not\equiv 0 \pmod 3$. We then use the identity $2 - 2e^{\frac{\pi i p_3 M}{3}} - e^{-\frac{\pi i p_3 M}{3}} + e^{\frac{2\pi i p_3 M}{3}} = 0$ to deduce (3.27) from (3.28). The proof for $N \equiv 3 \pmod 6$ is similar. □

4. Proof of Theorem 1.2

We are now in a position to prove Theorem 1.2. Again, we follow the order in which the Bailey pairs were introduced in Section 2.2.

Proof. We begin with the proof of (1.32). By Proposition 3.3, (1.32) is equivalent to

$$-\frac{1}{2}\sum_{n=0}^{6(6p+1)N}\chi(n)\left(1-\frac{n}{6(6p+1)N}\right)\zeta_N^{M\frac{n^2-(6p-1)^2}{24(6p+1)}} = H_p^{(6)}(\zeta_N^M) \tag{4.1}$$

where $\chi = \chi_{(2,3,6p+1)}^{(1,1,2p)}$ is the odd periodic function given by

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 42p+5,\ 54p+7,\ 54p+11,\ 66p+13 \pmod{72p+12}, \\ -1 & \text{if } n \equiv 6p-1,\ 18p+1,\ 18p+5,\ 30p+7 \pmod{72p+12}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

By Theorem 2.7,

$$H_p^{(6)}(q) = \sum_{k=0}^{N-1}(-1)^k q^{\binom{k+1}{2}+(p-1)(k^2+k)}\alpha_k \tag{4.3}$$

where $q=\zeta_N^M$ and (α_k,β_k) is the Bailey pair in (2.16). Next, we claim that

$$4\sum_{k=0}^{N-1}(-1)^k q^{\binom{k+1}{2}+(p-1)(k^2+k)}\alpha_k = -\sum_{n=0}^{6(6p+1)N}\chi(n)q^{\frac{n^2-(6p-1)^2}{24(6p+1)}} \tag{4.4}$$

when $q=\zeta_N^M$. To deduce (4.4), we first demonstrate that (as polynomials in q)

$$\sum_{k=0}^{N-1}(-1)^k q^{\binom{k+1}{2}+(p-1)(k^2+k)}\alpha_k = -\sum_{n=0}^{2(6p+1)N}\chi(n)q^{\frac{n^2-(6p-1)^2}{24(6p+1)}}. \tag{4.5}$$

We proceed by induction on N. The induction step consists of verifying the identity

$$(-1)^N q^{\binom{N+1}{2}+(p-1)(N^2+N)}\alpha_N = -\sum_{n=0}^{12p+1}\chi((12p+2)(N+1)-n)q^{\frac{((12p+2)(N+1)-n)^2-(6p-1)^2}{24(6p+1)}}. \tag{4.6}$$

We consider three cases. First suppose that $N=3r$. If r is even, then

$$\chi((12p+2)(N+1)-n) = \chi(12p+2-n)$$

and so the right-hand side of (4.6) is

$$\begin{aligned} -\sum_{n=0}^{12p+1}\chi(12p+2-n)q^{\frac{((12p+2)(N+1)-n)^2-(6p-1)^2}{24(6p+1)}} &= -\chi(6p-1)q^{\frac{((12p+2)(N+1)-(6p+3))^2-(6p-1)^2}{24(6p+1)}} \\ &= q^{\frac{((12p+2)(N+1)-(6p+3))^2-(6p-1)^2}{24(6p+1)}} \end{aligned} \tag{4.7}$$

by (4.2). Similarly, if r is odd, then we have

$$\chi((12p+2)(N+1)-n) = \chi((12p+2)(N+3)-24p-4-n) = -\chi(n+24p+4),$$

and so the right-hand side of (4.6) is

$$\begin{aligned}\sum_{n=0}^{12p+1} \chi(n+24p+4) q^{\frac{((12p+2)(N+1)-n)^2-(6p-1)^2}{24(6p+1)}} &= \chi(30p+7) q^{\frac{((12p+2)(N+1)-(6p+3))^2-(6p-1)^2}{24(6p+1)}} \\ &= -q^{\frac{((12p+2)(N+1)-(6p+3))^2-(6p-1)^2}{24(6p+1)}}\end{aligned} \tag{4.8}$$

by (4.2). In both (4.7) and (4.8), one can check using (2.16) that the signs and powers of q on both sides of (4.6) match. Next suppose that $N = 3r+1$. If r is even, then

$$\chi((12p+2)(N+1)-n) = \chi(24p+4-n)$$

and so the right-hand side of (4.6) is

$$\begin{aligned}&-\sum_{n=0}^{12p+1} \chi(24p+4-n) q^{\frac{((12p+2)(N+1)-n)^2-(6p-1)^2}{24(6p+1)}} \\ &\quad = -\chi(18p+1) q^{\frac{((12p+2)(N+1)-(6p+3))^2-(6p-1)^2}{24(6p+1)}} - \chi(18p+5) q^{\frac{((12p+2)(N+1)-(6p-1))^2-(6p-1)^2}{24(6p+1)}} \\ &\quad = q^{\frac{((12p+2)(N+1)-(6p+3))^2-(6p-1)^2}{24(6p+1)}} + q^{\frac{((12p+2)(N+1)-(6p-1))^2-(6p-1)^2}{24(6p+1)}}\end{aligned} \tag{4.9}$$

by (4.2). Similarly, if r is odd, then we have

$$\chi((12p+2)(N+1)-n) = \chi((12p+2)(N+2)-12p-2-n) = -\chi(n+12p+2),$$

and so the right-hand side of (4.6) is

$$\begin{aligned}&\sum_{n=0}^{12p+1} \chi(n+12p+2) q^{\frac{((12p+2)(N+1)-n)^2-(6p-1)^2}{24(6p+1)}} \\ &\quad = \chi(18p+5) q^{\frac{((12p+2)(N+1)-(6p+3))^2-(6p-1)^2}{24(6p+1)}} + \chi(18p+1) q^{\frac{((12p+2)(N+1)-(6p-1))^2-(6p-1)^2}{24(6p+1)}} \\ &\quad = -q^{\frac{((12p+2)(N+1)-(6p+3))^2-(6p-1)^2}{24(6p+1)}} - q^{\frac{((12p+2)(N+1)-(6p-1))^2-(6p-1)^2}{24(6p+1)}}\end{aligned} \tag{4.10}$$

by (4.2). In both (4.9) and (4.10), one can check using (2.16) that the signs and powers of q on both sides of (4.6) match. As the argument is similar for $N = 3r-1$, we omit the details.

Now, by (4.5), it suffices to prove

$$\sum_{n=0}^{6(6p+1)N} \chi(n) q^{\frac{n^2-(6p-1)^2}{24(6p+1)}} = 4 \sum_{n=0}^{2(6p+1)N} \chi(n) q^{\frac{n^2-(6p-1)^2}{24(6p+1)}} \tag{4.11}$$

when $q = \zeta_N^M$. This follows by taking $p_3 = 6p+1$ and multiplying both sides of (3.26) by $q^{-\frac{(6p-1)^2}{24(6p+1)}}$ in Corollary 3.6. Thus, (4.5) with $q = \zeta_N^M$ and (4.11) imply (4.4). By Lemma 3.1, we have

$$\sum_{n=0}^{6(6p+1)N} \chi(n) q^{\frac{n^2-(6p-1)^2}{24(6p+1)}} = 2q^{-\frac{(6p-1)^2}{24(6p+1)}} \sum_{n=0}^{6(6p+1)N} \chi(n) \left(1 - \frac{n}{6(6p+1)N}\right) q^{\frac{n^2}{24(6p+1)}} \tag{4.12}$$

where $q = \zeta_N^M$. Combining (4.3), (4.4) and (4.12), we arrive at

$$2\sum_{n=0}^{6(6p+1)N} \chi(n)\left(1-\frac{n}{6(6p+1)N}\right)\zeta_N^{M\frac{n^2-(6p-1)^2}{24(6p+1)}} = -4H_p^{(6)}(\zeta_N^M). \tag{4.13}$$

Multiplying both sides of (4.13) by $-\frac{1}{4}$ yields (4.1). Thus, (1.32) follows.

As the proofs for (1.26)–(1.31) and (1.33)–(1.36) are similar to that of (1.32), we sketch the details. One first uses Proposition 3.3 to express the limiting value of the relevant Eichler integral as a finite sum (as in the left-hand side of (4.1)). Next, by Theorems 2.8–2.16, we can express each of the left-hand sides of (1.26)–(1.31) and (1.33)–(1.36) as a finite sum involving α_k, $(q^{p(k^2+2k)}\alpha_k)^*$ or $\widehat{\alpha}_k$ (analogous to (4.3)). One then proves an underlying identity between polynomials in q akin to (4.5). Here we list these identities as significant care is required in some cases. As for (4.5), the proofs proceed by induction on N. For (1.33), the identity is

$$\sum_{k=0}^{N-1}(-1)^k q^{\binom{k+1}{2}+(p-1)(k^2+k)}\alpha_k = -\sum_{n=0}^{2(6p+1)N}\chi(n)q^{\frac{n^2-(6p+5)^2}{24(6p+1)}} \tag{4.14}$$

where (α_k, β_k) is the Bailey pair (2.17) and $\chi = \chi_{(2,3,6p+1)}^{(1,1,2p+1)}$. For (1.29), it is

$$\sum_{k=0}^{N-1}(-1)^k q^{\binom{k+1}{2}+(p-1)(k^2+k)}\alpha_k = -\sum_{n=0}^{2(6p-1)N}\chi(n)q^{\frac{n^2-(6p-5)^2}{24(6p-1)}} \tag{4.15}$$

where (α_k, β_k) is the Bailey pair (2.18) and $\chi = \chi_{(2,3,6p-1)}^{(1,1,2p-1)}$. For (1.30), it is

$$\sum_{k=0}^{N-1}(-1)^k q^{\binom{k+1}{2}+(p-1)(k^2+k)}\alpha_k = -\sum_{n=0}^{2(6p-1)N}\chi(n)q^{\frac{n^2-(6p+1)^2}{24(6p-1)}} \tag{4.16}$$

where (α_k, β_k) is the Bailey pair (2.19) and $\chi = \chi_{(2,3,6p-1)}^{(1,1,2p)}$. For (1.34), it is

$$(1-q)\sum_{k=0}^{N-1}(-1)^k q^{\binom{k+1}{2}+(p-1)(k^2+k)}\alpha_k = \sum_{n=0}^{2(6p+1)N}\chi(n)q^{\frac{n^2-(6p-5)^2}{24(6p+1)}} \tag{4.17}$$

where (α_k, β_k) is the Bailey pair (2.20) and $\chi = \chi_{(2,3,6p+1)}^{(1,1,1)}$. For (1.26), it is

$$q^{-1}(1-q)\sum_{k=0}^{N-1}(-1)^k q^{\binom{k+1}{2}+(p-1)(k^2+k)}\alpha_k = \sum_{n=0}^{2(6p-1)N}\chi(n)q^{\frac{n^2-(6p+5)^2}{24(6p-1)}} \tag{4.18}$$

for $p \geq 2$ and

$$q^{-1}(1-q)\sum_{k=0}^{N-1}(-1)^k q^{\binom{k+1}{2}}\alpha_k = \sum_{n=0}^{10N}\chi(n)q^{\frac{n^2-11^2}{120}} - q^{-1}\lambda_N \tag{4.19}$$

where

$$\lambda_N = \lambda_N(q) := -2 + \begin{cases} (-1)^r q^{\frac{15r^2+r}{2}} + (-1)^r q^{\frac{15r^2-r}{2}} & \text{if } N = 3r, \\ (-1)^r q^{\frac{15r^2+11r+2}{2}} & \text{if } N = 3r+1, \\ (-1)^{r+1} q^{\frac{15r^2+19r+6}{2}} & \text{if } N = 3r+2 \end{cases}$$

for $p=1$. Here, (α_k,β_k) is the Bailey pair (2.21) and $\chi=\chi^{(1,1,1)}_{(2,3,6p-1)}$. One can check that if $q=\zeta_N^M$, then $\lambda_N=\lambda_N(\zeta_N^M)=-1$. For (1.31), it is

$$\sum_{k=0}^{N-1}(-1)^k q^{-\binom{k+1}{2}}(q^{p(k^2+2k)}\alpha_k)^* = -\sum_{n=0}^{2(6p-1)N}\chi(n)q^{\frac{n^2-(12p-5)^2}{24(6p-1)}} \tag{4.20}$$

where (α_k,β_k) is the Bailey pair (2.22) and $\chi=\chi^{(1,1,3p-1)}_{(2,3,6p-1)}$. For (1.35), it is

$$\sum_{k=0}^{N-1}(-1)^k q^{-\binom{k+1}{2}}(q^{p(k^2+2k)}\alpha_k)^* = -\sum_{n=0}^{2(6p+1)N}\chi(n)q^{\frac{n^2-(12p-1)^2}{24(6p+1)}} \tag{4.21}$$

where (α_k,β_k) is the Bailey pair (2.23) and $\chi=\chi^{(1,1,3p)}_{(2,3,6p+1)}$. For (1.28), it is

$$\begin{aligned}\sum_{k=0}^{N-1}(-1)^k q^{-\binom{k+1}{2}+p(k+1)^2}\widehat{\alpha}_k + 1 + \frac{1}{2}(-1)^N q^{-\binom{N}{2}+pN^2}(\widehat{\alpha}_{N-1}-\gamma_{N-1})&\\ = -\sum_{n=0}^{2(6p-1)N}\chi(n)q^{\frac{n^2-1}{24(6p-1)}}&\end{aligned} \tag{4.22}$$

where (α_k,β_k) is the Bailey pair (2.24),

$$\gamma_k=\gamma_k(q):=\begin{cases} q^{3r^2-2r}+q^{3r^2-r} & \text{if } k=3r-1,\\ -q^{3r^2+r} & \text{if } k=3r,\\ -q^{3r^2+2r} & \text{if } k=3r+1\end{cases}$$

and $\chi=\chi^{(1,1,p)}_{(2,3,6p-1)}$. Again, one can confirm that if $q=\zeta_N^M$, then $\gamma_{N-1}=\gamma_{N-1}(\zeta_N^M)=-1$ and $(-1)^N q^{-\binom{N}{2}+pN^2}=-1$. Finally, for (1.36), it is

$$\begin{aligned}\sum_{k=0}^{N-2}(-1)^k q^{-\binom{k+1}{2}+p(k+1)^2}\widehat{\alpha}_k - \frac{1}{2}(-1)^N q^{-\binom{N}{2}+pN^2}\widehat{\alpha}_{N-1} - 1 - \frac{1}{2}(-1)^N q^{pN^2}\kappa_N&\\ = -\sum_{n=0}^{2(6p+1)N}\chi(n)q^{\frac{n^2-1}{24(6p+1)}}&\end{aligned} \tag{4.23}$$

where (α_k,β_k) is the Bailey pair (2.25),

$$\kappa_N=\kappa_N(q):=\begin{cases} -q^{\frac{r(3r-1)}{2}}-q^{\frac{r(3r+1)}{2}} & \text{if } N=3r,\\ q^{\frac{r(3r+1)}{2}} & \text{if } N=3r+1,\\ q^{\frac{(r+1)(3r+2)}{2}} & \text{if } N=3r+2\end{cases}$$

and $\chi=\chi^{(1,1,p)}_{(2,3,6p+1)}$. In this case, if $q=\zeta_N^M$, then $\kappa_N=\kappa_N(\zeta_N^M)=(-1)^{N+1}$. We now take $q=\zeta_N^M$ in each of (4.14)–(4.23), simplify and then combine with statements similar to (4.11) (resulting from Corollary 3.5) to establish versions of (4.4). After applying Lemma 3.1 (to obtain variants of (4.12)), the results follow by combining steps and multiplying by an appropriate factor. In total, this yields (1.26)–(1.31) and (1.33)–(1.36). □

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