TAME KERNELS AND FURTHER 4-RANK DENSITIES

ROBERT OSBURN AND BRIAN MURRAY

ABSTRACT. There has been recent progress on computing the 4-rank of the tame kernel $K_2(\mathcal{O}_F)$ for F a quadratic number field. For certain quadratic number fields, this progress has led to "density results" concerning the 4-rank of tame kernels. These results were first mentioned in [6] and proven in [8]. In this paper, we consider some additional quadratic number fields and obtain further density results of 4-ranks of tame kernels. Additionally, we give tables which might indicate densities in some generality.

1. INTRODUCTION

We are interested in the structure of the 2-Sylow subgroup of $K_2(\mathcal{O}_F)$. As $K_2(\mathcal{O}_F)$ is a finite abelian group, it is a product of cyclic groups of prime power order. We say the 2^j-rank, $j \geq 1$, of $K_2(\mathcal{O}_F)$ is the number of cyclic factors of $K_2(\mathcal{O}_F)$ of order divisible by 2^j. In [12], the 2-rank of the tame kernel is given by Tate's 2-rank formula. In the case where F is a quadratic number field, Browkin and Schinzel in [3] simplified the 2-rank formula. In their formula, we can determine the 2-rank by counting the number of elements in $\{\pm 1, \pm 2\}$ which are norms from the given quadratic field and the number of odd primes which are ramified in the given quadratic field. Now what about 4-ranks of $K_2(\mathcal{O}_F)$?

In [6], Conner and Hurrelbrink characterize the 4-rank of $K_2(\mathcal{O})$ for certain quadratic number fields in terms of positive definite binary quadratic forms. This characterization led to a connection between densities of certain sets of primes and 4-rank values. Specifically, the author in [8] considers the 4-rank of $K_2(\mathcal{O})$ for the quadratic number fields $\mathbb{Q}(\sqrt{pl})$, $\mathbb{Q}(\sqrt{2pl})$, $\mathbb{Q}(\sqrt{-pl})$, $\mathbb{Q}(\sqrt{-2pl})$ for primes $p \equiv 7 \bmod 8$, $l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = 1$. In [6], it was shown that for the fields $E = \mathbb{Q}(\sqrt{pl})$, $\mathbb{Q}(\sqrt{2pl})$ and $F = \mathbb{Q}(\sqrt{-pl})$, $\mathbb{Q}(\sqrt{-2pl})$,

$$\text{4-rank } K_2(\mathcal{O}_E) = 1 \text{ or } 2,$$

$$\text{4-rank } K_2(\mathcal{O}_F) = 0 \text{ or } 1.$$

The idea in [8] is to fix $p \equiv 7 \bmod 8$ and consider the set

$$\Omega = \{l \text{ rational prime} : l \equiv 1 \bmod 8 \text{ and } \left(\frac{l}{p}\right) = \left(\frac{p}{l}\right) = 1\}.$$

In [8], the following was proved.

Theorem 1.1. *For the fields $\mathbb{Q}(\sqrt{pl})$ and $\mathbb{Q}(\sqrt{2pl})$, 4-rank 1 and 2 each appear with natural density $\frac{1}{2}$ in Ω. For the fields $\mathbb{Q}(\sqrt{-pl})$ and $\mathbb{Q}(\sqrt{-2pl})$, 4-rank 0 and 1 each appear with natural density $\frac{1}{2}$ in Ω.*

In this paper, we consider the 4-rank of $K_2(\mathcal{O})$ for the quadratic number fields $\mathbb{Q}(\sqrt{pl})$, $\mathbb{Q}(\sqrt{-pl})$ for primes $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = 1$ and $\mathbb{Q}(\sqrt{pl})$ for primes $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = -1$. We will see that for the primes $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = 1$,

$$4\text{-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 1 \text{ or } 2,$$
$$4\text{-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-pl})}) = 1 \text{ or } 2.$$

For the primes $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = -1$, we will see

$$4\text{-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 0 \text{ or } 1.$$

Let us fix a prime $p \equiv 1 \bmod 8$ and consider the sets

$$A = \{l \text{ rational prime} : l \equiv 1 \bmod 8 \text{ and } \left(\frac{l}{p}\right) = 1\},$$

$$B = \{l \text{ rational prime} : l \equiv 1 \bmod 8 \text{ and } \left(\frac{l}{p}\right) = -1\}.$$

The goal of this paper is to prove two theorems analogous to Theorem 1.1, namely:

Theorem 1.2. *For the field $\mathbb{Q}(\sqrt{pl})$, 4-rank 1 and 2 appear with natural density $\frac{3}{4}$ and $\frac{1}{4}$ in A. For the field $\mathbb{Q}(\sqrt{-pl})$, 4-rank 1 and 2 each appear with natural density $\frac{1}{2}$ in A.*

Theorem 1.3. *For the field $\mathbb{Q}(\sqrt{pl})$, 4-rank 0 and 1 each appear with natural density $\frac{1}{2}$ in B.*

Now for squarefree, odd integers d, consider the sets

$$X = \{d : d = pl\}$$

and

$$Y = \{d : d = -pl\}$$

for distinct primes p and l.

We have computed the following: For $15 \leq d < 10^6$, there are 168331 d's in X. Among them, there are 35787 d's (21.26%) yielding 4-rank 0, 128468 d's (76.32%) yielding 4-rank 1, and 4076 d's (2.42%) yielding 4-rank 2.

For $-10^6 < d \leq -15$, there are 168330 d's in Y. Among them, there are 104056 d's (61.82%) yielding 4-rank 0, 63054 d's (37.46%) yielding 4-rank 1, and 1220 d's (.72%) yielding 4-rank 2. As a consequence of Theorems 1.2, 1.3 and Tables I and II in [9] and [10], we obtain:

Corollary 1.4. *For the fields* $\mathbb{Q}(\sqrt{pl})$, *4-rank 0, 1, and 2 appear with natural density* $\frac{13}{64}$, $\frac{97}{128}$, $\frac{5}{128}$ *respectively in* X.

Corollary 1.5. *For the fields* $\mathbb{Q}(\sqrt{-pl})$, *4-rank 0, 1, and 2 appear with natural density* $\frac{37}{64}$, $\frac{13}{32}$, *and* $\frac{1}{64}$ *respectively in* Y.

2. Preliminaries

Let $\mathcal{D}$ be a Galois extension of $\mathbb{Q}$, and $G = Gal(\mathcal{D}/\mathbb{Q})$. Let $Z(G)$ denote the center of G and $\mathcal{D}^{Z(G)}$ denote the fixed field of $Z(G)$. Let p be a rational prime which is unramified in $\mathcal{D}$ and β be a prime of $\mathcal{D}$ containing p. Let $\left(\frac{\mathcal{D}/\mathbb{Q}}{p}\right)$ denote the Artin symbol of p and $\{g\}$ the conjugacy class containing one element $g \in G$. In Sections 5 and 6 we use the following elementary lemma from [8].

Lemma 2.1. $\left(\frac{\mathcal{D}/\mathbb{Q}}{p}\right) = \{g\}$ *for some* $g \in Z(G)$ *if and only if* p *splits completely in* $\mathcal{D}^{Z(G)}$.

Thus if we can show that rational primes split completely in the fixed field of the center of a certain Galois group G, then we know the associated Artin symbol is a conjugacy class containing one element. Note that determining the order of $Z(G)$ gives us the number of possible choices for the Artin symbol. The order of $Z(G)$ can be computed using the following setup.

Let G_1 and G_2 be finite groups and A a finite abelian group. Suppose $r_1 : G_1 \to A$ and $r_2 : G_2 \to A$ are two epimorphisms and $\mathcal{G} \subset G_1 \times G_2$ is the set $\{(g_1, g_2) \in G_1 \times G_2 : r_1(g_1) = r_2(g_2)\}$. Since A is abelian, there is an epimorphism $r : G_1 \times G_2 \to A$ given by $r(g_1, g_2) = r_1(g_1)r_2(g_2)^{-1}$. Thus $\mathcal{G} = ker(r) \subset G_1 \times G_2$. One can check that $Z(\mathcal{G}) = \mathcal{G} \cap Z(G_1 \times G_2)$. From [8], we provide:

Lemma 2.2. $Z(\mathcal{G}) = Z(G_1) \times Z(G_2) \iff r_1\big|_{Z(G_1)}$ *and* $r_2\big|_{Z(G_2)}$ *are both trivial.*

We will use the following definition throughout this paper.

Definition 2.3. For primes $p \equiv l \equiv 1 \mod 8$ with $\left(\frac{l}{p}\right) = 1$, $\mathcal{K} = \mathbb{Q}(\sqrt{2p})$, and $h^+(\mathcal{K})$ the narrow class number of $\mathcal{K}$, we say:

l satisfies $\langle 1, 32 \rangle$ if and only if $l = x^2 + 32y^2$ for some $x, y \in \mathbb{Z}$

l satisfies $\langle p, -2 \rangle$ if and only if $l^{\frac{h^+(\mathcal{K})}{4}} = pn^2 - 2m^2$ for some $n, m \in \mathbb{Z}$ with $m \not\equiv 0 \mod l$

l satisfies $\langle 1, -2p \rangle$ if and only if $l^{\frac{h^+(\mathcal{K})}{4}} = n^2 - 2pm^2$ for some $n, m \in \mathbb{Z}$ with $m \not\equiv 0 \mod l$.

3. First Extension

Consider the fixed prime $p \equiv 1 \mod 8$. Note p splits completely in $\mathcal{L} = \mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$ and so

$$p\mathcal{O}_\mathcal{L} = \mathfrak{B}\mathfrak{B}'$$

for some primes $\mathfrak{B} \neq \mathfrak{B}'$ in $\mathcal{L}$. The field $\mathcal{L}$ has narrow class number $h^+(\mathcal{L}) = 1$ as $h(\mathcal{L}) = 1$ and $N_{\mathcal{L}/\mathbb{Q}}(\epsilon) = -1$ where $\epsilon = 1 + \sqrt{2}$ is a fundamental unit of $\mathbb{Q}(\sqrt{2})$. Similar to Lemma 2.1 in [6],

Lemma 3.1. *The prime $\mathfrak{B}$ which occurs in the decomposition of $p\mathcal{O}_\mathcal{L}$ has a generator $\pi = a + b\sqrt{2} \in \mathcal{O}_\mathcal{L}$, unique up to a sign and to multiplication by the square of a unit in $\mathcal{O}_\mathcal{L}^*$ for which $N_{\mathcal{L}/\mathbb{Q}}(\pi) = a^2 - 2b^2 = p$.*

The degree 4 extension $\mathbb{Q}(\sqrt{2}, \sqrt{\pi})$ over $\mathbb{Q}$ has normal closure $\mathbb{Q}(\sqrt{2}, \sqrt{\pi}, \sqrt{p})$ as $N_{\mathcal{L}/\mathbb{Q}}(\pi) = p$. Set

$$N = \mathbb{Q}(\sqrt{2}, \sqrt{\pi}, \sqrt{p}).$$

Then N is Galois over $\mathbb{Q}$ and $[N : \mathbb{Q}] = 8$. By Corollary 24.5 in [4], 4 divides the narrow class number of $\mathbb{Q}(\sqrt{2p})$. Moreover N over $\mathbb{Q}(\sqrt{2p})$ is unramified at all finite primes. Similar to Lemma 2.3 in [6], N is the unique unramified cyclic degree 4 extension over $\mathbb{Q}(\sqrt{2p})$.

Consider the rational primes $l \equiv 1 \mod 8$ for which $\left(\frac{l}{p}\right) = 1$. These primes split completely in $\mathbb{Q}(\sqrt{2}, \sqrt{p})$ over $\mathbb{Q}$. We characterize such primes l that split completely in N over $\mathbb{Q}$. As N is the unique unramified cyclic degree 4 extension of $\mathbb{Q}(\sqrt{2p})$, mimicing Lemma 3.3 in [6] yields

Lemma 3.2. *Let $l \equiv 1 \mod 8$ be a prime such that $\left(\frac{l}{p}\right) = 1$. Then:*

l splits completely in N if and only if l satisfies $\langle 1, -2p \rangle$.

Similar to Lemma 3.4 in [6], with 2 (respectively, $\mathfrak{D}$, the unique dyadic prime in $\mathcal{O}_{\mathbb{Q}(\sqrt{2p})}$) replaced by p (respectively $\mathfrak{p}$, the prime over p whose class is the unique element of order 2 in the narrow ideal class group of $\mathbb{Q}(\sqrt{2p})$), we obtain

Lemma 3.3. *Let $l \equiv 1 \bmod 8$ be a prime such that $\left(\frac{l}{p}\right) = 1$. Then: l does not split completely in N if and only if l satisfies $\langle p, -2 \rangle$.*

We now relate the characterizations of Lemmas 3.2 and 3.3 to the quadratic symbol $\left(\frac{\pi}{l}\right)$. From Lemma 3.1, we have a presentation $\pi = a + b\sqrt{2} \in \mathcal{O}_{\mathcal{L}}$ with $N_{\mathcal{L}/\mathbb{Q}}(\pi) = p$. Let $\mathfrak{P}$ be a prime above l in $\mathcal{O}_{\mathcal{L}}$. As l splits in $\mathcal{L}$ over $\mathbb{Q}$, then the residue field $\mathcal{O}_{\mathcal{L}}/\mathfrak{P}$ is isomorphic to $\mathbb{Z}/l\mathbb{Z} = \mathbb{F}_l$, the field with l elements. As 2 is a square modulo l, we have $2 \equiv \alpha^2 \bmod l$ for some $\alpha \in \mathbb{F}_l^*$. Thus we can identify $\pi = a + b\sqrt{2} \in \mathcal{O}_{\mathcal{L}}$ with $a + b\alpha \in \mathbb{F}_l$. When we write the symbol $\left(\frac{\pi}{l}\right)$, it is understood that we mean $\left(\frac{a+b\alpha}{l}\right)$. From the discussion in Section 3 of [6], the symbol $\left(\frac{\pi}{l}\right)$ is well defined and l splits completely in N over $\mathbb{Q}$ if and only if $\left(\frac{\pi}{l}\right) = 1$. Combining this discussion with Lemmas 3.2 and 3.3, we have:

Proposition 3.4. *Let $l \equiv 1 \bmod 8$ be a prime with $\left(\frac{l}{p}\right) = 1$. Then:*

$$l \text{ satisfies } \langle 1, -2p \rangle \iff \left(\frac{\pi}{l}\right) = 1,$$
$$l \text{ satisfies } \langle p, -2 \rangle \iff \left(\frac{\pi}{l}\right) = -1.$$

4. Matrices and Symbols

Hurrelbrink and Kolster [7] generalize Qin's approach in [9], [10] and obtain 4-rank results by computing $\mathbb{F}_2$-ranks of certain matrices of local Hilbert symbols. Let us be more specific. Let $F = \mathbb{Q}(\sqrt{d})$, $d \neq 0, 1$, squarefree. Let $p_1, p_2, \ldots, p_t$ denote the odd primes dividing d. Recall 2 is a norm from F $\iff$ all p_i's are $\equiv \pm 1 \bmod 8$. If so, then d is a norm from $\mathbb{Q}(\sqrt{2})$, thus

$$d = u^2 - 2w^2$$

for $u, w \in \mathbb{Z}$. Now consider two matrices:

If $d < 0$,

$$M'_{F/\mathbb{Q}} = \begin{pmatrix} (-d, p_1)_2 & (-d, p_1)_{p_1} & \cdots & (-d, p_1)_{p_t} \\ (-d, p_2)_2 & (-d, p_2)_{p_1} & \cdots & (-d, p_2)_{p_t} \\ \vdots & \vdots & & \vdots \\ (-d, p_{t-1})_2 & (-d, p_{t-1})_{p_1} & \cdots & (-d, p_{t-1})_{p_t} \\ (-d, v)_2 & (-d, v)_{p_1} & \cdots & (-d, v)_{p_t} \\ (-d, -1)_2 & (-d, -1)_{p_1} & \cdots & (-d, -1)_{p_t} \end{pmatrix}$$

If $d > 0$,

$$M_{F/\mathbb{Q}} = \begin{pmatrix} (-d, p_1)_2 & (-d, p_1)_{p_1} & \cdots & (-d, p_1)_{p_t} \\ (-d, p_2)_2 & (-d, p_2)_{p_1} & \cdots & (-d, p_2)_{p_t} \\ \vdots & \vdots & & \vdots \\ (-d, p_{t-1})_2 & (-d, p_{t-1})_{p_1} & \cdots & (-d, p_{t-1})_{p_t} \\ (-d, v)_2 & (-d, v)_{p_1} & \cdots & (-d, v)_{p_t} \\ (d, -1)_2 & (d, -1)_{p_1} & \cdots & (d, -1)_{p_t} \end{pmatrix}$$

If 2 is not a norm from F, set $v = 2$. Otherwise, set $v = u + w$. Replacing the 1's by 0's and the -1's by 1's, we calculate the matrix rank over $\mathbb{F}_2$. Why look at these matrices ? From [7],

Lemma 4.1. *Let $F = \mathbb{Q}(\sqrt{d})$, $d \neq 0, 1$, squarefree. Then*
(i)If $d < 0$, then 4-rank $K_2(\mathcal{O}_F) = t - rk\ (M'_{F/\mathbb{Q}})$
(ii) If $d > 0$, then 4-rank $K_2(\mathcal{O}_F) = t - rk\ (M_{F/\mathbb{Q}}) + a' - a$

where

$$a = \begin{cases} 0 & \text{if 2 is a norm from F} \\ 1 & \text{otherwise} \end{cases}$$

and

$$a' = \begin{cases} 0 & \text{if both -1 and 2 are norms from F} \\ 1 & \text{if exactly one of -1 or 2 is a norm from F} \\ 2 & \text{if none of -1 or 2 are norms from F.} \end{cases}$$

Recall that our cases are:

- $\mathbb{Q}(\sqrt{pl})$, $\mathbb{Q}(\sqrt{-pl})$ where $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = 1$,

- $\mathbb{Q}(\sqrt{pl})$ for $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = -1$.

In both cases 2 is a norm from $\mathbb{Q}(\sqrt{pl})$ and $\mathbb{Q}(\sqrt{-pl})$. Before we view the matrices for our cases, we characterize the symbol $(-d, v)_2$ for $d = pl, -pl$ (see Lemmas 5.3 and 5.15 in [7]).

- $(-pl, v)_2 = 1 \iff$ both p, l satisfy $\langle 1, 32 \rangle$ or neither p, l satisfy $\langle 1, 32 \rangle$,

- $(pl, v)_2 = 1$.

Also, v is an l-adic unit and hence

$$(-pl, v)_l = (l, v)_l = \left(\frac{v}{l}\right).$$

Similarly, $(-pl, v)_p = \left(\frac{v}{p}\right)$. In the entries of the matrices below, we write $(-pl, v)_2$, $\left(\frac{v}{l}\right)$, and $\left(\frac{v}{p}\right)$ remembering to first evalute the symbols, make the

substitutions 1 for 0 and -1 for 1, and then calculate the matrix rank over $\mathbb{F}_2$. Now what are the matrices in our situations?

- For $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = 1$, we have:

$$M_{\mathbb{Q}(\sqrt{pl})/\mathbb{Q}} = \begin{pmatrix} 0 & 0 & 0 \\ (-pl, v)_2 & \left(\frac{v}{p}\right) & \left(\frac{v}{l}\right) \\ 0 & 0 & 0 \end{pmatrix},$$

$$M'_{\mathbb{Q}(\sqrt{-pl})/\mathbb{Q}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \left(\frac{v}{p}\right) & \left(\frac{v}{l}\right) \\ 0 & 0 & 0 \end{pmatrix}.$$

- For $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = -1$, we have:

$$M_{\mathbb{Q}(\sqrt{pl})/\mathbb{Q}} = \begin{pmatrix} 0 & 1 & 1 \\ (-pl, v)_2 & \left(\frac{v}{p}\right) & \left(\frac{v}{l}\right) \\ 0 & 0 & 0 \end{pmatrix}.$$

Remark 4.2. For $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = 1$, we have:

- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 1 \iff \text{rank } M_{\mathbb{Q}(\sqrt{pl})/\mathbb{Q}} = 1 \iff (-pl, v)_2 = 1$, $\left(\frac{v}{l}\right) = -1$ or $(-pl, v)_2 = -1 \iff$ both p, l satisfy $\langle 1, 32 \rangle$, $\left(\frac{v}{l}\right) = -1$ or neither p, l satisfy $\langle 1, 32 \rangle$, $\left(\frac{v}{l}\right) = -1$, or exactly one of p, l satisifes $\langle 1, 32 \rangle$.

- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 2 \iff \text{rank } M_{\mathbb{Q}(\sqrt{pl})/\mathbb{Q}} = 0 \iff (-pl, v)_2 = 1$, $\left(\frac{v}{l}\right) = 1 \iff$ both p, l satisfy $\langle 1, 32 \rangle$, $\left(\frac{v}{l}\right) = 1$ or neither p, l satisfy $\langle 1, 32 \rangle$, $\left(\frac{v}{l}\right) = 1$.

- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-pl})}) = 1 \iff \text{rank } M'_{\mathbb{Q}(\sqrt{-pl})/\mathbb{Q}} = 1 \iff \left(\frac{v}{l}\right) = -1$.

- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-pl})}) = 2 \iff \text{rank } M'_{\mathbb{Q}(\sqrt{-pl})/\mathbb{Q}} = 0 \iff \left(\frac{v}{l}\right) = 1$.

Remark 4.3. For $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = -1$:

- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 1 \iff \text{rank } M_{\mathbb{Q}(\sqrt{pl})/\mathbb{Q}} = 1 \iff (-pl, v)_2 = 1 \iff$ both p, l satisfy $\langle 1, 32 \rangle$ or neither p, l satisfy $\langle 1, 32 \rangle$.

- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 0 \iff \text{rank } M_{\mathbb{Q}(\sqrt{pl})/\mathbb{Q}} = 2 \iff (-pl, v)_2 = -1 \iff$ exactly one of p, l satisfies $\langle 1, 32 \rangle$.

We can now prove Theorem 1.3.

Proof. Consider the sets

$$\mathcal{A}_1 = \{l \text{ prime: } l \equiv 1 \bmod 8 \text{ and } l \text{ satisfies } \langle 1, 32 \rangle \},$$

$$\mathcal{A}_2 = \{l \text{ prime: } l \equiv 1 \bmod 8 \text{ and } l \text{ does not satisfy } \langle 1, 32 \rangle \}.$$

By the discussion before Corollary 24.2 in [4], $\mathcal{A}_1$ and $\mathcal{A}_2$ each have density $\frac{1}{2}$ in the set of all primes $l \equiv 1 \bmod 8$. By Dirichlet's Theorem on primes in arithmetic progessions, $\mathcal{A}_1$ and $\mathcal{A}_2$ each have density $\frac{1}{8}$ in the set of all primes l. Note that for primes $p \equiv 1 \bmod 8$, the sets

$$\mathcal{B}_1 = \{l \text{ prime: } l \equiv 1 \bmod 8, \left(\tfrac{l}{p}\right) = -1, \text{ and } l \text{ satisfies } \langle 1, 32 \rangle \},$$

$$\mathcal{B}_2 = \{l \text{ prime: } l \equiv 1 \bmod 8, \left(\tfrac{l}{p}\right) = -1, \text{ and } l \text{ does not satisfy } \langle 1, 32 \rangle \}.$$

each have density $\frac{1}{2}$ in $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. Thus $\mathcal{B}_1$ and $\mathcal{B}_2$ have densities $\frac{1}{16}$ in the set of all primes l. If p satisfies $\langle 1, 32 \rangle$, then by Remark 4.3:

$$\mathcal{B}_1 = \{l \text{ prime: } l \equiv 1 \bmod 8, \left(\tfrac{l}{p}\right) = -1, \text{ and } 4\text{-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 1 \},$$

$$\mathcal{B}_2 = \{l \text{ prime: } l \equiv 1 \bmod 8, \left(\tfrac{l}{p}\right) = -1, \text{ and } 4\text{-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 0\}.$$

For each $\mathcal{B}_i$, $i = 1, 2$, we have:

$$\left\{ \begin{array}{c} \text{Density of } \mathcal{B}_i \text{ in the} \\ \text{set of all primes } l \end{array} \right\} = \left\{ \begin{array}{c} \text{Density of} \\ \mathcal{B}_i \text{ in } B \end{array} \right\} \cdot \left\{ \begin{array}{c} \text{Density of } B \text{ in the} \\ \text{set of all primes } l \end{array} \right\}$$

where B has density $\frac{1}{8}$ in the set of all primes l. Thus 4-rank 0 and 4-rank 1 each appear with natural density $\frac{1}{2}$ in B. A similar argument works if p does not satisfy $\langle 1, 32 \rangle$.

$\square$

For the primes $p \equiv l \equiv 1 \bmod 8$ with $\left(\tfrac{l}{p}\right) = 1$, let us relate the Legendre symbol $\left(\tfrac{v}{l}\right)$ to the quadratic symbol $\left(\tfrac{\pi}{l}\right)$. For primes $l \equiv 1 \bmod 8$, the quadratic symbol $\left(\tfrac{1+\sqrt{2}}{l}\right)$ is well defined and satisfies, see [1],

$$\left(\tfrac{1+\sqrt{2}}{l}\right) = 1 \iff l \text{ satisfies } \langle 1, 32 \rangle.$$

Proposition 4.4. *Let $d = \pm pl$ be as above, $d = u^2 - 2w^2$ with $u, w \in \mathbb{Z}$. Then:*

$$\left(\tfrac{v}{l}\right) = \left(\tfrac{\pi}{l}\right)\left(\tfrac{1+\sqrt{2}}{l}\right) \text{ if } d = pl$$

$$\left(\tfrac{v}{l}\right) = \left(\tfrac{\pi}{l}\right) \text{ if } d = -pl.$$

Proof. From the proof of Proposition 4.6 in [6], we use the identity

$$\left(\tfrac{v}{l}\right) = \left(\tfrac{\gamma+\delta\sqrt{2}}{l}\right)\left(\tfrac{1+\sqrt{2}}{l}\right)$$

where $\frac{d}{l} = N_{\mathcal{L}/\mathbb{Q}}(\gamma + \delta\sqrt{2})$ for γ, $\delta \in \mathbb{Z}$. For $d = pl$, we have $\frac{d}{l} = p = N_{\mathcal{L}/\mathbb{Q}}(\pi)$ and thus $\gamma + \delta\sqrt{2} = \pi$, up to squares. For $d = -pl$, we have $\frac{d}{l} = -p = -N_{\mathcal{L}/\mathbb{Q}}(\pi)$ and so $\gamma + \delta\sqrt{2} = (1 + \sqrt{2})\pi$, up to squares.

$\square$

In view of Proposition 3.4, Remark 4.2, and Proposition 4.4, we can determine the 4-rank of the tame kernel in terms of quadratic forms.

Proposition 4.5. *For $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = 1$:*

- *4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 1 \iff$ both p, l satisfy $\langle 1, 32 \rangle$, l satisfies $\langle p, -2 \rangle$ or neither p, l satisfy $\langle 1, 32 \rangle$, l satisfies $\langle p, -2 \rangle$ or exactly one of p, l satisfies $\langle 1, 32 \rangle$*
- *4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 2 \iff$ both p, l satisfy $\langle 1, 32 \rangle$, l satisfies $\langle 1, -2p \rangle$ or neither p, l satisfy $\langle 1, 32 \rangle$, l satisfies $\langle 1, -2p \rangle$*
- *4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-pl})}) = 1 \iff l$ satisfies $\langle p, -2 \rangle$*
- *4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-pl})}) = 2 \iff l$ satisfies $\langle 1, -2p \rangle$.*

It should be noted that Qin Yue has obtained characterizations of 4-rank values, similar to Proposition 4.5, by additionally assuming that the fundamental unit of $\mathbb{Q}(\sqrt{2p})$, $p \equiv 1 \bmod 8$, has norm -1, see [11].

5. Two Artin Symbols

5.1. **First Artin symbol.** Consider $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$. Let $\epsilon = 1 + \sqrt{2} \in (\mathbb{Z}[\sqrt{2}])^*$. Then ϵ is a fundamental unit of $\mathbb{Q}(\sqrt{2})$ which has norm -1. The degree 4 extension $\mathbb{Q}(\sqrt{2}, \sqrt{\epsilon})$ over $\mathbb{Q}$ has normal closure $\mathbb{Q}(\sqrt{2}, \sqrt{\epsilon}, \sqrt{-1})$. Set

$$N_1 = \mathbb{Q}(\sqrt{2}, \sqrt{\epsilon}, \sqrt{-1}).$$

Note that $Gal(N_1/\mathbb{Q})$ is the dihedral group of order 8 and $Z(Gal(N_1/\mathbb{Q})) = Gal(N_1/\mathbb{Q}(\sqrt{2}, \sqrt{-1}))$ (see [8], Section 3.2).

Only the prime 2 ramifies in $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{\epsilon})$, and so only the prime 2 ramifies in the compositum N_1 over $\mathbb{Q}$. Now as $l \in A$ is unramified in N_1 over $\mathbb{Q}$, the Artin symbol $\left(\frac{N_1/\mathbb{Q}}{\beta}\right)$ is defined for primes β of $\mathcal{O}_{N_1}$ containing l. Let $\left(\frac{N_1/\mathbb{Q}}{l}\right)$ denote the conjugacy class of $\left(\frac{N_1/\mathbb{Q}}{\beta}\right)$ in $Gal(N_1/\mathbb{Q})$. The primes $l \in A$ split completely in $\mathbb{Q}(\sqrt{2}, \sqrt{-1})$ and $N_1^{Z(Gal(N_1/\mathbb{Q}))} = \mathbb{Q}(\sqrt{2}, \sqrt{-1})$. Thus by Lemma 2.1, we have that $\left(\frac{N_1/\mathbb{Q}}{l}\right) = \{g\}$ for some $g \in Z(Gal(N_1/\mathbb{Q}))$. As $Z(Gal(N_1/\mathbb{Q}))$ has order 2, there are two possible

choices for $\left(\frac{N_1/\mathbb{Q}}{l}\right)$. Combining this statement with Addendum (3.7) from [6], we have

Remark 5.1.

$$\left(\frac{N_1/\mathbb{Q}}{l}\right) = \{id\} \quad \Longleftrightarrow \quad l \text{ splits completely in } N_1$$
$$\Longleftrightarrow \quad l \text{ satisfies } \langle 1, 32 \rangle.$$

5.2. **Second Artin symbol.** In section 3, we considered

$$N = \mathbb{Q}(\sqrt{2}, \sqrt{\pi}, \sqrt{p}),$$

the unique unramified cyclic degree 4 extension over $\mathbb{Q}(\sqrt{2p})$. Similar to the extension N_1, we have $Gal(N/\mathbb{Q})$ is the dihedral group of order 8 and $Z(Gal(N/\mathbb{Q})) = Gal(N/\mathbb{Q}(\sqrt{2}, \sqrt{p}))$.

Proposition 5.2. *If $l \in A$, then l is unramified in N over $\mathbb{Q}$.*

Proof. Since $p \equiv 1 \bmod 8$, the discriminant of $\mathbb{Q}(\sqrt{2p})$ is $8p$. For $l \in A$, we have $\left(\frac{2p}{l}\right) = 1$ and so l is unramified in $\mathbb{Q}(\sqrt{2p})$. We conclude that l is unramified in N over $\mathbb{Q}$. $\qquad\qquad\square$

As $l \in A$ is unramified in N over $\mathbb{Q}$, the Artin symbol $\left(\frac{N/\mathbb{Q}}{\beta}\right)$ is defined for primes β of $\mathcal{O}_N$ containing l. Let $\left(\frac{N/\mathbb{Q}}{l}\right)$ denote the conjugacy class of $\left(\frac{N/\mathbb{Q}}{\beta}\right)$ in $Gal(N/\mathbb{Q})$. The primes $l \in A$ split completely in $\mathbb{Q}(\sqrt{2}, \sqrt{p})$ and $N^{Z(Gal(N/\mathbb{Q}))} = \mathbb{Q}(\sqrt{2}, \sqrt{p})$. By Lemma 2.1, we have that $\left(\frac{N/\mathbb{Q}}{l}\right) = \{h\}$ for some $h \in Z(Gal(N/\mathbb{Q}))$. As $Z(Gal(N/\mathbb{Q}))$ has order 2, there are two possible choices for $\left(\frac{N/\mathbb{Q}}{l}\right)$. Combining this statement and Lemmas 3.2 and 3.3, we have

Remark 5.3.

$$\left(\frac{N/\mathbb{Q}}{l}\right) = \{id\} \quad \Longleftrightarrow \quad l \text{ splits completely in } N$$
$$\Longleftrightarrow \quad l \text{ satisfies } \langle 1, -2p \rangle.$$

$$\left(\frac{N/\mathbb{Q}}{l}\right) \neq \{id\} \quad \Longleftrightarrow \quad l \text{ does not split completely in } N$$
$$\Longleftrightarrow \quad l \text{ satisfies } \langle p, -2 \rangle.$$

6. A COMPOSITE AND PROOF OF THEOREM 1.2

In this section we consider the composite field $N_1 N$. Set

$$\mathfrak{N} = N_1 N.$$

Note that $[\mathfrak{N} : \mathbb{Q}] = 32$. As N_1 and N are normal extensions of $\mathbb{Q}$, $\mathfrak{N}$ is a normal extension of $\mathbb{Q}$.

For $l \in A$, l is unramified in $\mathfrak{N}$ as it is unramified in N_1 and N. The Artin symbol $\left(\frac{\mathfrak{N}/\mathbb{Q}}{\beta} \right)$ is now defined for some prime β of $\mathcal{O}_{\mathfrak{N}}$ containing l. Let $\left(\frac{\mathfrak{N}/\mathbb{Q}}{l} \right)$ denote the conjugacy class of $\left(\frac{\mathfrak{N}/\mathbb{Q}}{\beta} \right)$ in $Gal(\mathfrak{N}/\mathbb{Q})$. Letting $M = \mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{p}) \subset \mathfrak{N}$, we prove

Lemma 6.1. $Z(Gal(\mathfrak{N}/\mathbb{Q})) = Gal(\mathfrak{N}/M)$ *is elementary abelian of order 4.*

Proof. For $\sigma \in Gal(\mathfrak{N}/M)$, σ can only change the sign of $\sqrt{\epsilon}$ and $\sqrt{\pi}$ as $\epsilon \in M$. Since $\mathfrak{N} = M(\sqrt{\epsilon}, \sqrt{\pi})$, $Gal(\mathfrak{N}/M)$ is elementary abelian of order 4. Now consider the restrictions $r_1 : G_1 \to Gal(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ and $r_2 : G_2 \to Gal(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ where $G_1 = Gal(N_1/\mathbb{Q})$ and $G_2 = Gal(N/\mathbb{Q})$. Clearly $r_1 \big|_{Z(G_1)}$ and $r_1 \big|_{Z(G_2)}$ are both trivial. Then by Lemma 2.2, $Z(\mathcal{G})$ is elementary abelian of order 4 where $\mathcal{G} = Gal(\mathfrak{N}/\mathbb{Q})$. Thus $Z(Gal(\mathfrak{N}/\mathbb{Q})) = Gal(\mathfrak{N}/M)$. $\square$

Now for $l \in A$, l splits completely in $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{2}, \sqrt{p})$ and so splits completely in the composite field $M = \mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{p})$. From Lemma 6.1, $\mathfrak{N}^{Z(Gal(\mathfrak{N}/\mathbb{Q}))} = \mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{p})$. So by Lemma 2.1, we have

$$\left(\frac{\mathfrak{N}/\mathbb{Q}}{l} \right) = \{k\} \text{ for some } k \in Gal(\mathfrak{N}/\mathbb{Q}).$$

As $Z(Gal(\mathfrak{N}/\mathbb{Q}))$ has order 4, there are four possible choices for $\left(\frac{\mathfrak{N}/\mathbb{Q}}{l} \right)$. Using Remarks 5.1 and 5.3, we now make the following one to one correspondences.

Remark 6.2. (i) $\left(\frac{\mathfrak{N}/\mathbb{Q}}{l} \right) = \{id\} \iff l$ splits completely in $\mathfrak{N} \iff$
$\left\{ \begin{array}{l} l \text{ splits completely in} \\ N_1 \text{ and } N \end{array} \right\} \iff \left\{ \begin{array}{l} l \text{ satisfies } \langle 1, 32 \rangle \\ l \text{ satisfies } \langle 1, -2p \rangle \end{array} \right\}$.

(ii) $\left(\frac{\mathfrak{N}/\mathbb{Q}}{l} \right) \neq \{id\} \iff l$ does not split completely in $\mathfrak{N}$. Now there are three cases:

(1) $\left\{ \begin{array}{l} l \text{ splits completely in } N_1, \\ \text{but does not in } N \end{array} \right\} \iff \left\{ \begin{array}{l} l \text{ satisfies } \langle 1, 32 \rangle \\ l \text{ satisfies } \langle p, -2 \rangle \end{array} \right\}$

(2) $\left\{ \begin{array}{l} l \text{ splits completely in } N \\ \text{but does not in } N_1 \end{array} \right\} \iff \left\{ \begin{array}{l} l \text{ does not satisfy } \langle 1, 32 \rangle \\ l \text{ satisfies } \langle 1, -2p \rangle \end{array} \right\}$

$$(3) \quad \left\{ \begin{array}{l} l \text{ does split completely} \\ \text{in } N_1 \text{ or } N \end{array} \right\} \iff \left\{ \begin{array}{l} l \text{ does not satisfy } \langle 1, 32 \rangle \\ l \text{ satisfies } \langle p, -2 \rangle \end{array} \right\}.$$

We can now prove Theorem 1.2

Proof. Consider the set $X = \{l \text{ prime} : l \text{ is unramified in } \mathfrak{N} \text{ and} \left(\frac{\mathfrak{N}/\mathbb{Q}}{l} \right) = \{k\}\}$ for some $k \in \text{Gal}(\mathfrak{N}/\mathbb{Q})$. By the Čebotarev Density Theorem, the set X has natural density $\frac{1}{32}$ in the set of all primes l. Recall

$$A = \{l \text{ rational prime} : l \equiv 1 \bmod 8 \text{ and } \left(\frac{l}{p} \right) = 1\}$$

for some fixed prime $p \equiv 1 \bmod 8$. By Dirichlet's Theorem on primes in arithmetic progressions, A has natural density $\frac{1}{8}$ in the set of all primes l. Thus X has natural density $\frac{1}{4}$ in A. If p satisfies $\langle 1, 32 \rangle$, then by Proposition 4.5,

$$4\text{-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 1 \iff \left\{ \begin{array}{l} l \text{ satisfies } \langle 1, 32 \rangle \\ l \text{ satisfies } \langle p, -2 \rangle \end{array} \right\} \text{ or } \left\{ \begin{array}{l} l \text{ does not} \\ \text{satisfy } \langle 1, 32 \rangle \end{array} \right\}$$

and

$$4\text{-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 2 \iff \left\{ \begin{array}{l} l \text{ satisfies } \langle 1, 32 \rangle \\ l \text{ satisfies } \langle 1, -2p \rangle \end{array} \right\}.$$

Using Remark 6.2, we see that for $\mathbb{Q}(\sqrt{pl})$, 4-rank 1 and 4-rank 2 appear with natural density $\frac{1}{4} + \frac{1}{2} = \frac{3}{4}$ and $\frac{1}{4}$ respectively. A similar argument works if p does not satisfy $\langle 1, 32 \rangle$. For $\mathbb{Q}(\sqrt{-pl})$, use Proposition 4.5 and Remark 6.2 to obtain that 4-rank 1 and 2 each appear with natural density $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ in A. $\qquad\square$

7. Proof of Two Corollaries

For squarefree, odd integers d, recall the sets $X = \{d : d = pl\}$ and $Y = \{d : d = -pl\}$ for distinct primes p and l. Now consider the sets

$$X_i = \{d : d = pl, \ p \equiv i \bmod 8\},$$
$$Y_i = \{d : -pl, \ p \equiv i \bmod 8\}.$$

Thus $X = X_1 \cup X_3 \cup X_5 \cup X_7$ and $Y = Y_1 \cup Y_3 \cup Y_5 \cup Y_7$. Additionally consider the sets

$$X_{i,j} = \{d : d = pl, \ p \equiv i \bmod 8, \ l \equiv j \bmod 8\},$$
$$Y_{i,j} = \{d : d = -pl, \ p \equiv i \bmod 8, \ l \equiv j \bmod 8\}.$$

Thus, for example, $X_1 = X_{1,1} \cup X_{1,3} \cup X_{1,5} \cup X_{1,7}$ and $Y_7 = Y_{7,1} \cup Y_{7,3} \cup Y_{7,5} \cup Y_{7,7}$.

In Tables 1 and 2 below, for $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})})$, we provide cases in which densities of 4-rank values follow from congruence conditions on p and l, a condition on the Legendre symbol $\left(\frac{l}{p} \right)$ (if any), and Dirichlet's theorem on

primes in arithmetic progressions. In Tables 3 and 4, we provide the same information for $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-pl})})$ (compare with [5] or Tables I and II in [9] and [10]).

TABLE 1: $\mathbb{Q}(\sqrt{pl})$

$p, l \bmod 8$	4-rank	Densities
3, 3	0	$\frac{1}{4}$ in X_3
5, 5	1	$\frac{1}{4}$ in X_5
7, 7	1	$\frac{1}{4}$ in X_7
3, 5	1	$\frac{1}{4}$ in X_3 and X_5
3, 7	1	$\frac{1}{4}$ in X_3 and X_7
5, 7	1	$\frac{1}{4}$ in X_5 and X_7

TABLE 2: $\mathbb{Q}(\sqrt{pl})$

p, l mod 8	Legendre symbols	4-rank	Densities
1, 3	$\left(\frac{l}{p}\right) = -1$	0	$\frac{1}{8}$ in X_1 and X_3
	$\left(\frac{l}{p}\right) = 1$	1	$\frac{1}{8}$ in X_1 and X_3
1, 5	$\left(\frac{l}{p}\right) = -1$	0	$\frac{1}{8}$ in X_1 and X_5
	$\left(\frac{l}{p}\right) = 1$	1	$\frac{1}{8}$ in X_1 and X_5
1, 7	$\left(\frac{l}{p}\right) = -1$	1	$\frac{1}{8}$ in X_1 and X_7
	$\left(\frac{l}{p}\right) = 1$	1	$\frac{1}{16}$ in X_1 and X_7
		2	$\frac{1}{16}$ in X_1 and X_7

TABLE 3: $\mathbb{Q}(\sqrt{-pl})$

p,l mod 8	4-rank	Densities
3, 3	1	$\frac{1}{4}$ in Y_3
5, 5	1	$\frac{1}{4}$ in Y_5
7, 7	1	$\frac{1}{4}$ in Y_7
3, 5	0	$\frac{1}{4}$ in Y_3 and Y_5
3, 7	0	$\frac{1}{4}$ in Y_3 and Y_7
5, 7	0	$\frac{1}{4}$ in Y_5 and Y_7

TABLE 4: $\mathbb{Q}(\sqrt{-pl})$

p, l mod 8	Legendre symbols	4-rank	Densities
1, 1	$\left(\frac{l}{p}\right) = -1$	1	$\frac{1}{8}$ in Y_1.
1, 3	$\left(\frac{l}{p}\right) = -1$	0	$\frac{1}{8}$ in Y_1 and Y_3
	$\left(\frac{l}{p}\right) = 1$	1	$\frac{1}{8}$ in Y_1 and Y_3
1, 5	$\left(\frac{l}{p}\right) = -1$	0	$\frac{1}{8}$ in Y_1 and Y_5
	$\left(\frac{l}{p}\right) = 1$	1	$\frac{1}{8}$ in Y_1 and Y_5
1, 7	$\left(\frac{l}{p}\right) = -1$	0	$\frac{1}{8}$ in Y_1 and Y_7
	$\left(\frac{l}{p}\right) = 1$	0	$\frac{1}{16}$ in Y_1 and Y_7
		1	$\frac{1}{16}$ in Y_1 and Y_7

Remark 7.1. By Theorem 1.2, $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = 1$ yields 4-rank 1 and 2 with densities $\frac{3}{32}$ and $\frac{1}{32}$ respectively in X_1. By Theorem 1.3, $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = -1$ yields 4-rank 0 and 1 each with density $\frac{1}{16}$ in X_1. We can now prove Corollary 1.4.

Proof. Regarding the set X_1:

- 4-rank 0, 1, and 2 appear with natural densities $\frac{1}{16}$, $\frac{3}{32} + \frac{1}{16} = \frac{5}{32}$, and $\frac{1}{32}$ in $X_{1,1}$
- 4-rank 0 and 1 each appear with natural densities $\frac{1}{8}$ in $X_{1,3}$
- 4-rank 0 and 1 each appear with natural densities $\frac{1}{8}$ in $X_{1,5}$
- 4-rank 1 and 2 appear with natural densities $\frac{1}{8} + \frac{1}{16} = \frac{3}{16}$ and $\frac{1}{16}$ in $X_{1,7}$.

Thus 4-rank 0, 1, and 2 appear with natural densities $\frac{5}{16}$, $\frac{19}{32}$, and $\frac{3}{32}$ in X_1. For the set X_3:

- 4-rank 0 and 1 each appear with natural density $\frac{1}{8}$ in $X_{3,1}$
- 4-rank 0 appears with natural density $\frac{1}{4}$ in $X_{3,3}$
- 4-rank 1 appears with natural density $\frac{1}{4}$ in $X_{3,5}$
- 4-rank 1 appears with natural density $\frac{1}{4}$ in $X_{3,7}$.

So 4-rank 0 and 1 appear with natural densities $\frac{3}{8}$ and $\frac{5}{8}$ in X_3. Similarly, 4-rank 0 and 1 appear with natural densities $\frac{1}{8}$ and $\frac{7}{8}$ in X_5 and 4-rank 1 and 2 appear with natural densities $\frac{15}{16}$ and $\frac{1}{16}$ in X_7. As each X_i has density $\frac{1}{4}$ in X,

- 4-rank 0 appears with natural density $\frac{5}{64} + \frac{3}{32} + \frac{1}{32} = \frac{13}{64}$ in X
- 4-rank 1 appears with natural density $\frac{19}{128} + \frac{5}{32} + \frac{7}{32} + \frac{15}{64} = \frac{97}{128}$ in X
- 4-rank 2 appears with natural density $\frac{3}{128} + \frac{1}{64} = \frac{5}{128}$ in X.

$\square$

Remark 7.2. By Theorem 1.2, $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = 1$ yields 4-rank 1 and 2 each with density $\frac{1}{16}$ in Y_1. We can now prove Corollary 1.5.

Proof. Regarding the set Y_1:

- 4-rank 1 and 2 appear with natural densities $\frac{1}{8} + \frac{1}{16} = \frac{3}{16}$ and $\frac{1}{16}$ in $Y_{1,1}$
- 4-rank 0 and 1 each appear with natural densities $\frac{1}{8}$ in $Y_{1,3}$
- 4-rank 0 and 1 each appear with natural densities $\frac{1}{8}$ in $Y_{1,5}$
- 4-rank 0 and 1 appear with natural densities $\frac{1}{8}$ and $\frac{1}{16} + \frac{1}{16} = \frac{1}{8}$ in $Y_{1,7}$.

Thus 4-rank 0, 1, and 2 appear with natural densities $\frac{3}{8}$, $\frac{9}{16}$, and $\frac{1}{16}$ in Y_1. For the set Y_3:

- 4-rank 0 and 1 each appear with natural density $\frac{1}{8}$ in $Y_{3,1}$
- 4-rank 1 appears with natural density $\frac{1}{4}$ in $Y_{3,3}$
- 4-rank 0 appears with natural density $\frac{1}{4}$ in $Y_{3,5}$
- 4-rank 0 appears with natural density $\frac{1}{4}$ in $Y_{3,7}$.

So 4-rank 0 and 1 appear with natural densities $\frac{5}{8}$ and $\frac{3}{8}$ in Y_3. Similarly, 4-rank 0 and 1 appear with natural densities $\frac{5}{8}$ and $\frac{3}{8}$ in Y_5 and 4-rank 0 and 1 appear with natural densities $\frac{11}{16}$ and $\frac{5}{16}$ in Y_7. As each Y_i has density $\frac{1}{4}$ in Y,

- 4-rank 0 appears with natural density $\frac{3}{32} + \frac{5}{32} + \frac{5}{32} + \frac{11}{64} = \frac{37}{64}$ in Y
- 4-rank 1 appears with natural density $\frac{9}{64} + \frac{3}{32} + \frac{3}{32} + \frac{5}{64} = \frac{13}{32}$ in Y
- 4-rank 2 appears with natural density $\frac{1}{64}$ in Y.

$\square$

APPENDIX

The approach of Hurrelbrink and Kolster in [7] led us to write a program in GP/PARI [2] which generates the numerical values in Tables 5-8. The aim is to motivate possible density results for tame kernels of quadratic number fields. In Tables 5 and 6, p, l, and r are distinct odd primes. In Tables 7 and 8, d is odd and squarefree.

TABLE 5

Cardinality	$105 \leq d = plr < 10^6$	%
| 4-rank 0 |	8247	6.827
| 4-rank 1 |	92544	76.605
| 4-rank 2 |	20000	16.555
| 4-rank 3 |	16	.013

TABLE 6

Cardinality	$-10^6 < d = -plr \leq -105$	%
| 4-rank 0 |	67970	56.2633
| 4-rank 1 |	50147	41.5100
| 4-rank 2 |	2688	2.2250
| 4-rank 3 |	2	.0017

TABLE 7

Cardinality	$3 \le d < 10^6$	%
| 4-rank 0 |	93736	23.1284
| 4-rank 1 |	278138	68.6278
| 4-rank 2 |	33148	8.1789
| 4-rank 3 |	263	.0649

TABLE 8

Cardinality	$-10^6 < d \le -3$	%
| 4-rank 0 |	251884	62.14985
| 4-rank 1 |	148669	36.68258
| 4-rank 2 |	4730	1.16708
| 4-rank 3 |	2	.00049

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Department of Mathematics & Statistics, McMaster University, 1280 Main Street West, Hamilton, Ontario L8S 4K1

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803

E-mail address: osburnr@icarus.math.mcmaster.ca

E-mail address: murray@math.lsu.edu