GENERALIZED RANK DEVIATIONS FOR OVERPARTITIONS

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Abstract. We prove formulas for generalized rank deviations for overpartitions. These formulas are in terms of Appell–Lerch series and sums of quotients of theta functions and extend work of Lovejoy and the second author. As an application, we compute a dissection.

1. Introduction

A partition of a natural number n is a non-increasing sequence of positive integers whose sum is n. An overpartition of n is a partition in which the first occurrence of each distinct part may be overlined. For example, the partitions and overpartitions of 4 are

$$4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1$$

and

$$4,\ \overline{4},\ 3+1,\ \overline{3}+1,\ 3+\overline{1},\ \overline{3}+\overline{1},\ 2+2, \overline{2}+2, 2+1+1,\ \overline{2}+1+1,\ 2+\overline{1}+1,$$
$$\overline{2}+\overline{1}+1,\ 1+1+1+1,\ \overline{1}+1+1+1,$$

respectively. For a curated survey which highlights the importance of overpartitions in q-series, number theory and algebraic combinatorics, see [7]. The rank of a partition λ is the largest part $\ell(\lambda)$ minus the number of parts $\#(\lambda)$. For overpartitions, one can consider two statistics, the rank or the M_2-rank which is defined by [14]

$$M_2\text{-rank}\,(\pi) = \left\lceil \frac{\ell(\pi)}{2} \right\rceil - \#(\pi) + \#(\pi_o) - \chi(\pi)$$

where π_o is the subpartition consisting of the odd non-overlined parts and

$$\chi(\pi) := \chi(\text{the largest part of } \pi \text{ is odd and non-overlined}).$$

Throughout, we use the standard notation $\chi(X) := 1$ if X is true and 0 if X is false. A recent topic of interest for these combinatorial objects and their enumerative data is the study of rank deviations [11, 15]. For integers $0 \leq a \leq M$ where $M \geq 2$, consider

$$\overline{D}(a, M) := \sum_{n \geq 0} \left(\overline{N}(a, M, n) - \frac{\overline{p}(n)}{M} \right) q^n \tag{1.1}$$

and

$$\overline{D}_2(a, M) := \sum_{n \geq 0} \left(\overline{N}_2(a, M, n) - \frac{\overline{p}(n)}{M} \right) q^n \tag{1.2}$$

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where $\overline{N}(a, M, n)$ denotes the number of overpartitions of n with rank congruent to a modulo M, $\overline{N}_2(a, M, n)$ denotes the number of overpartitions of n with M_2-rank congruent to a modulo M and $\overline{p}(n)$ is the number of overpartitions of n. Some impetuses for finding explicit formulas for the rank deviations (1.1) and (1.2) are to correct the literature [15, Remark 1.6], provide a general framework in which one can recover all known rank difference identities for overpartitions [15, Section 4] and prove the modularity of rank generating functions in arithmetic progressions (e.g., [11, Section 7]) thereby generalizing [6, 8, 16].

For an integer $d \geq 1$, consider the generating function

$$\begin{aligned} \mathcal{O}_d(z;q) &:= \frac{(-q)_\infty}{(q)_\infty}\left(1 + 2\sum_{n\geq 1}\frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+dn}}{(1-zq^{dn})(1-z^{-1}q^{dn})}\right) \\ &=: \sum_{\substack{m\in\mathbb{Z}\\ n\geq 0}} \overline{N}_d(m,n) z^m q^n \end{aligned} \tag{1.3}$$

where

$$(x)_\infty = (x;q)_\infty := \prod_{k=0}^{\infty}(1-xq^k).$$

The $d = 1$ case of (1.3) gives the two-variable generating function for the rank for overpartitions [13] while the $d = 2$ case yields the two-variable generating function for the M_2-rank for overpartitions [14]. The mock and quantum modular properties of (1.3) were the focus of [9, 12]. Motivated by [9, 12, 15], our aim is to find explicit formulas for the generalized rank deviations for overpartitions

$$\overline{D}_d(a, M) := \sum_{n\geq 0}\left(\overline{N}_d(a,M,n) - \frac{\overline{p}(n)}{M}\right)q^n \tag{1.4}$$

where

$$\overline{N}_d(a,M,n) := \sum_{k\equiv a \pmod{M}} \overline{N}_d(k,n). \tag{1.5}$$

To state our main results, we require some setup. Consider the Appell–Lerch series

$$m(x,q,z) := \frac{1}{j(z;q)}\sum_{r\in\mathbb{Z}}\frac{(-1)^r q^{\binom{r}{2}} z^r}{1-q^{r-1}xz}. \tag{1.6}$$

Here, x, q and z are non-zero complex numbers with $|q| < 1$, neither z nor xz is an integral power of q and

$$j(z;q) := (z)_\infty (q/z)_\infty (q)_\infty = \sum_{n\in\mathbb{Z}}(-1)^n q^{\binom{n}{2}} z^n. \tag{1.7}$$

Let

$$\Delta(x, z_1, z_0; q) := \frac{z_0 J_1^3 j(z_1/z_0;q) j(xz_0z_1;q)}{j(z_0;q)j(z_1;q)j(xz_0;q)j(xz_1;q)} \tag{1.8}$$

and

$$\Psi_k^n(x,z,z';q) := -\frac{x^k z^{k+1} J_{n^2}^3}{j(z;q)j(z';q^{n^2})}\sum_{t=0}^{n-1}\frac{q^{\binom{t+1}{2}+kt}(-z)^t j(-q^{\binom{n+1}{2}+nk+nt}(-z)^n/z', q^{nt}(xz)^n z'; q^{n^2})}{j(-q^{\binom{n}{2}-nk}(-x)^n z', q^{nt}(xz)^n; q^{n^2})} \tag{1.9}$$

where $J_m := (q^m;q^m)_\infty$ and $j(z_1,z_2;q) := j(z_1;q)j(z_2;q)$. We use the term "generic" to mean that the parameters do not cause poles in the Appell–Lerch series or in the quotients of theta functions. Finally, for odd $d \geq 1$ and generic z, z_0 and $z' \in \mathbb{C}^*$, let

$$\Lambda(d,z,z_0,z') := (-1)^{\frac{d+1}{2}} q^{-\frac{(d-1)^2}{4}} z^{\frac{d-1}{d}} \Big(\Psi^d_{\frac{d-1}{2}}(z^{-\frac{2}{d}}q^d, z_0, z'; q^2) + \frac{1}{d}\sum_{t=0}^{d-1} \zeta_d^{-t} \Delta(\zeta_d^{-2t} z^{-\frac{2}{d}} q^d, \zeta_d^t z^{\frac{1}{d}} q^{-\frac{d-1}{2}}, z_0; q^2)\Big). \tag{1.10}$$

Here and throughout, ζ_M denotes a primitive root of unity of order M.

Theorem 1.1. *Let $d \geq 1$ be an odd integer and $2 \leq a \leq M$. For generic z', z'' and $z_0 \in \mathbb{C}^*$, we have the following generating functions:*

(i) If a and M are even, then

$$\begin{aligned} \overline{D}_d(a,M) + \overline{D}_d(a-1,M) \;=\;& \chi(a=M) \\ &+ 2(-1)^{\frac{a}{2}} q^{-\frac{a^2}{4}+\frac{a}{2}(1-d^2)} m((-1)^{\frac{M}{2}+1} q^{\frac{M^2}{4}-\frac{aM}{2}+\frac{M}{2}(1-d^2)}, q^{\frac{M^2}{2}}, z') \\ &- 2q^{-d^2}\Psi^{\frac{M}{2}}_{\frac{a}{2}-1}(q^{-d^2},-1,z';q^2) \\ &+ \frac{2}{M}\sum_{j=1}^{M} \zeta_{\frac{M}{2}}^{-\frac{aj}{2}} (1-\zeta_M^j)\Lambda(d,\zeta_M^j,z_0,-1). \end{aligned} \tag{1.11}$$

(ii) If a is even and M is odd, then

$$\begin{aligned} \overline{D}_d(a,M) + \overline{D}_d(a-1,M) \;=\;& 2(-1)^{\frac{a}{2}} q^{-d^2(\frac{2M-a}{2})^2} m(q^{d^2M(a-M)}, q^{2d^2M^2}, z') \\ &+ 2(-1)^{\frac{M+1-a}{2}} q^{-d^2(\frac{M+1-a}{2})^2} m(q^{d^2M(a-1)}, q^{2d^2M^2}, z'') \\ &- 2\Psi^M_{\frac{2M-a}{2}}(q^{d^2},-1,z';q^{2d^2}) + 2\Psi^M_{\frac{M+1-a}{2}}(q^{d^2},-1,z'';q^{2d^2}) \\ &+ \frac{2}{M}\sum_{j=1}^{M-1} \zeta_M^{-aj}(1-\zeta_M^j)\Lambda(d,\zeta_M^j,z_0,-1). \end{aligned} \tag{1.12}$$

(iii) If a is odd and M is odd, then

$$\begin{aligned} \overline{D}_d(a,M) + \overline{D}_d(a-1,M) \;=\;& \chi(a=M) \\ &- 2(-1)^{\frac{M-a}{2}} q^{-d^2(\frac{M-a}{2})^2} m(q^{d^2Ma}, q^{2d^2M^2}, z') \\ &+ 2(-1)^{\frac{a+1}{2}} q^{-d^2(\frac{2M-a+1}{2})^2} m(q^{d^2M(a-M-1)}, q^{2d^2M^2}, z'') \\ &- 2\Psi^M_{\frac{M-a}{2}}(q^{d^2},-1,z';q^{2d^2}) + 2\Psi^M_{\frac{2M+1-a}{2}}(q^{d^2},-1,z'';q^{2d^2}) \\ &+ \frac{2}{M}\sum_{j=1}^{M-1} \zeta_M^{-aj}(1-\zeta_M^j)\Lambda(d,\zeta_M^j,z_0,-1). \end{aligned} \tag{1.13}$$

Theorem 1.2. *Let $d \geq 2$ be an even integer and $1 \leq a \leq M-1$. For generic z', $z'' \in \mathbb{C}^*$, we have the generating function*

$$\begin{aligned}\overline{D}_d(a,M)+\overline{D}_d(a-1,M) \;=\;& \chi(a=1)+2(-1)^{\frac{da}{2}}q^{-\frac{d^2a^2}{4}}m((-1)^{1+\frac{dM}{2}}q^{\frac{d^2}{4}(M^2-2Ma)},q^{\frac{d^2M^2}{2}},z')\\&+\;2(-1)^{\frac{d}{2}(a-1)+1}q^{-\frac{d^2}{4}(a^2-2a+1)}m((-1)^{1+\frac{dM}{2}}q^{\frac{d^2}{4}(M^2-2M(a-1))},q^{\frac{d^2M^2}{2}},z'')\\&+\;2\Psi_a^M((-1)^{\frac{d}{2}+1}q^{\frac{d^2}{4}},-1,z';q^{\frac{d^2}{2}})-2\Psi_{a-1}^M((-1)^{\frac{d}{2}+1}q^{\frac{d^2}{4}},-1,z'';q^{\frac{d^2}{2}})\\&+\;\frac{2}{M}(-1)^{\frac{d}{2}}q^{-\frac{d^2}{4}}\sum_{j=1}^{M-1}\zeta_M^{j-aj}(1-\zeta_M^j)\Psi_0^{\frac{d}{2}}(\zeta_M^{\frac{2j}{d}}q^{1-d},q,-1;q^2).\end{aligned}\tag{1.14}$$

Remark 1.3. As discussed in Section 4, there is no loss in generality in computing pairs of deviations. Theorems 1.1 and 1.2 can be used to find a formula for any single $\overline{D}_d(a,M)$.

Remark 1.4. From [12, page 1154] and (1.5), we have

$$\overline{N}_d(a,M,n)=\overline{N}_d(M-a,M,n)$$

and so by (1.4)

$$\overline{D}_d(a,M)=\overline{D}_d(M-a,M).\tag{1.15}$$

If d and a are odd and M is even, then $\overline{D}_d(a,M)+\overline{D}_d(a-1,M)$ is computed from Theorem 1.1 (i) using the fact that

$$\overline{D}_d(a,M)+\overline{D}_d(a-1,M)=\overline{D}_d(M-a+1,M)+\overline{D}_d(M-a,M)$$

which follows from (1.15).

Remark 1.5. Let $n \in \mathbb{Z}$. If we take $d=1$ and $z_0=-1$ in Theorem 1.1 and use

$$j(q^n;q)=0,\tag{1.16}$$

then one recovers [15, Theorem 1.1]. If we let $d=2$ in Theorem 1.2 and employ [10, Eqs. (2.2a) and (2.2b)]

$$j(q^nx;q)=(-1)^nq^{-\binom{n}{2}}x^{-n}j(x;q)\tag{1.17}$$

and

$$j(x;q)=j(q/x;q)=-xj(x^{-1};q),\tag{1.18}$$

then we obtain [15, Theorem 1.2].

The paper is organized as follows. In Section 2, we recall the required background on Appell–Lerch series and prove two crucial formulas (see Proposition 2.4) which express a normalized version of $\mathcal{O}_d(z;q)$ in terms of Appell–Lerch series and (1.10) if d is odd and in terms of Appell–Lerch series and (1.9) if d is even. We also give some other key properties of (1.7) and establish 3-dissections of certain eta quotients, both of which will be beneficial in Section 5. In Section 3, we prove Theorems 1.1 and 1.2. In Section 4, we clarify Remark 1.3. In Section 5, we provide an application of Theorem 1.1 by computing the 3-dissection of $\mathcal{O}_3(\zeta_3;q)$ (cf. [12, Theorem 1.3]).

Finally, we comment that the combinatorial interpretation of $\overline{N}_d(m,n)$ involves a certain weighted count of buffered Frobenius representations [17, Theorem 1.3]. It is still an open problem to find a proper combinatorial definition of the "M_d-rank" for overparitions [17, Section 6].

2. Preliminaries

2.1. **Appell–Lerch series and $\overline{S}_d(z;q)$.** We first state two key results concerning Appell–Lerch series. The first relates two such series with different generic parameters z_1 and z_0 [10, Theorem 3.3] while the second is an orthogonality result [10, Theorem 3.9]. Recall (1.6), (1.8) and (1.9).

Lemma 2.1. *For generic x, z_0 and $z_1 \in \mathbb{C}^*$,*

$$m(x,q,z_1) - m(x,q,z_0) = \Delta(x,z_1,z_0;q). \tag{2.1}$$

Lemma 2.2. *Let n and k be integers with $0 \leq k < n$. Then*

$$\sum_{t=0}^{n-1} \zeta_n^{-kt} m(\zeta_n^t x, q, z) = nq^{-\binom{k+1}{2}}(-x)^k m(-q^{\binom{n}{2}-nk}(-x)^n, q^{n^2}, z') + n\Psi_k^n(x,z,z';q). \tag{2.2}$$

We also require [10, Eqs. (3.2b), (3.2e) and (3.3)]

$$m(x,q,z) = x^{-1}m(x^{-1},q,z^{-1}), \tag{2.3}$$

$$m(x,q,z) = x^{-1} - x^{-1}m(qx,q,z), \tag{2.4}$$

$$m(q,q^2,-1) = \frac{1}{2} \tag{2.5}$$

and [10, Eq. (4.7)]

$$\frac{1}{j(q;q^2)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{n^2+n}}{1-xq^n} = -x^{-1}m(x^{-2}q,q^2,x). \tag{2.6}$$

In addition, we often use the following well-known fact. Let $s \in \mathbb{Z}$. Then

$$\sum_{j=0}^{n-1} \zeta_n^{sj} = \begin{cases} n & \text{if } s \equiv 0 \pmod{n}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

To explicitly compute the generalized rank deviations (1.4), we use the following formula. As the proof is similar to that of [15, Proposition 2.3], we omit it. Let

$$\overline{S}_d(z;q) := (1+z)\mathcal{O}_d(z;q). \tag{2.8}$$

Proposition 2.3. *We have*

$$\frac{1}{M}\sum_{j=1}^{M-1} \zeta_M^{-aj}\overline{S}_d(\zeta_M^j;q) = \overline{D}_d(a,M) + \overline{D}_d(a-1,M). \tag{2.9}$$

We now need to express $\overline{S}_d(z;q)$ in terms of Appell–Lerch series and sums of theta quotients. Recall (1.10).

Proposition 2.4. *Let z, z_0 and $z' \in \mathbb{C}^*$ be generic. For d odd, we have*

$$\overline{S}_d(z;q) = (1-z)\left(1 - 2m(z^{-2}q^{d^2}, q^{2d^2}, z') + 2\Lambda(d,z,z_0,z')\right). \tag{2.10}$$

For d even, we have

$$\overline{S}_d(z;q) = (1-z)\left(-1 + 2m((-1)^{\frac{d}{2}+1}zq^{\frac{d^2}{4}}, q^{\frac{d^2}{2}}, z') + 2(-1)^{\frac{d}{2}}zq^{-\frac{d^2}{4}}\Psi_0^{\frac{d}{2}}(z^{\frac{2}{d}}q^{1-d}, q, z'; q^2)\right). \tag{2.11}$$

Proof. By (1.3), the fact that

$$\frac{1}{(1-zq^{dn})(1-z^{-1}q^{dn})}=\frac{-z^2}{(1-z^2)(1-zq^{dn})}+\frac{1}{(1-z^2)(1-z^{-1}q^{dn})}$$

and some simplifications, we have

$$\mathcal{O}_d(z;q)=\frac{1-z}{1+z}\left(1+\frac{2z}{j(q;q^2)}\sum_{n\in\mathbb{Z}}\frac{(-1)^nq^{n^2+dn}}{1-zq^{dn}}\right). \tag{2.12}$$

For d odd, we let $n\to n-\frac{d-1}{2}$ in (2.12) to obtain

$$\mathcal{O}_d(z;q)=\frac{1-z}{1+z}\left(1+\frac{2z(-1)^{\frac{d-1}{2}}q^{-\frac{(d-1)^2}{4}-\frac{d-1}{2}}}{j(q;q^2)}\sum_{n\in\mathbb{Z}}\frac{(-1)^nq^{n^2+n}}{1-zq^{dn-\frac{d(d-1)}{2}}}\right). \tag{2.13}$$

An application of

$$\frac{1}{1-x^d}=\frac{1}{d}\sum_{t=0}^{d-1}\frac{1}{1-\zeta_d^tx} \tag{2.14}$$

to (2.13) yields

$$\mathcal{O}_d(z;q)=\frac{1-z}{1+z}\left(1+\frac{2z(-1)^{\frac{d-1}{2}}q^{-\frac{(d-1)^2}{4}-\frac{d-1}{2}}}{dj(q;q^2)}\sum_{t=0}^{d-1}\sum_{n\in\mathbb{Z}}\frac{(-1)^nq^{n^2+n}}{1-\zeta_d^tz^{\frac{1}{d}}q^{-\frac{d-1}{2}+n}}\right). \tag{2.15}$$

Taking $x=\zeta_d^tz^{\frac{1}{d}}q^{-\frac{d-1}{2}}$ in (2.6) turns (2.15) into

$$\mathcal{O}_d(z;q)=\frac{1-z}{1+z}\left(1-\frac{2z(-1)^{\frac{d-1}{2}}q^{-\frac{(d-1)^2}{4}}z^{-\frac{1}{d}}}{d}\sum_{t=0}^{d-1}\zeta_d^{-t}m(\zeta_d^{-2t}z^{-\frac{2}{d}}q^d,q^2,\zeta_d^tz^{\frac{1}{d}}q^{-\frac{d-1}{2}})\right). \tag{2.16}$$

We now let $x=\zeta_d^{-2t}z^{-\frac{2}{d}}q^d$, $q\to q^2$ and $z_1=\zeta_d^tz^{\frac{1}{d}}q^{-\frac{d-1}{2}}$ in (2.1) to express (2.16) as

$$\begin{aligned}\mathcal{O}_d(z;q)=\frac{1-z}{1+z}\Bigg(1-\frac{2z(-1)^{\frac{d-1}{2}}q^{-\frac{(d-1)^2}{4}}z^{-\frac{1}{d}}}{d}\Bigg(&\sum_{t=0}^{d-1}\zeta_d^{-t}m(\zeta_d^{-2t}z^{-\frac{2}{d}}q^d,q^2,z_0)\\&+\sum_{t=0}^{d-1}\zeta_d^{-t}\Delta(\zeta_d^{-2t}z^{-\frac{2}{d}}q^d,\zeta_d^tz^{\frac{1}{d}}q^{-\frac{d-1}{2}},z_0;q^2)\Bigg)\Bigg).\end{aligned} \tag{2.17}$$

Finally, we apply (2.2) with $n=d$, $k=\frac{d-1}{2}$, $q\to q^2$ and $x=z^{-\frac{2}{d}}q^d$ to (2.17), simplify and appeal to (1.10). By (2.8), this yields (2.10).

For d even, we let $n\to n-\frac{d}{2}$ in (2.12) to obtain

$$\mathcal{O}_d(z;q)=\frac{1-z}{1+z}\left(1+\frac{2z(-1)^{\frac{d}{2}}q^{-\frac{d^2}{4}}}{j(q;q^2)}\sum_{n\in\mathbb{Z}}\frac{(-1)^nq^{n^2}}{1-zq^{dn-\frac{d^2}{2}}}\right). \tag{2.18}$$

Using (2.14), (2.18) becomes

$$\mathcal{O}_d(z;q)=\frac{1-z}{1+z}\left(1+\frac{4z(-1)^{\frac{d}{2}}q^{\frac{-d^2}{4}}}{d}\sum_{t=0}^{\frac{d}{2}-1}m(\zeta_{\frac{d}{2}}^{t}z^{\frac{2}{d}}q^{1-d},q^2,q)\right). \tag{2.19}$$

Applying (2.2) with $n=\frac{d}{2}$, $k=0$, $q\to q^2$ and $x=z^{\frac{2}{d}}q^{1-d}$ to (2.19) yields

$$\mathcal{O}_d(z;q)=\frac{1-z}{1+z}\Bigg(1+2z(-1)^{\frac{d}{2}}q^{-\frac{d^2}{4}}m((-1)^{\frac{d}{2}+1}zq^{-\frac{d^2}{4}},q^{\frac{d^2}{2}},z')$$
$$+2(-1)^{\frac{d}{2}}zq^{-\frac{d^2}{4}}\Psi_0^{\frac{d}{2}}(z^{\frac{2}{d}}q^{1-d},q,z';q^2)\Bigg).$$

By (2.4) and (2.8), (2.11) follows. □

Remark 2.5. If we take $d=1$ and $z_0=z'=-1$ in (2.10), then use (1.10) combined with (1.18) and the $z_1=-1$, $z_0=z$ case of (2.1), we recover [15, Eq. (3.1)]. If we take $d=2$ and $z'=q$ in (2.11) and then use (1.9) and (1.16), we obtain [15, Eq. (3.3)].

2.2. **Identities with $j(z;q)$ and 3-dissections.** In order to facilitate an application of Theorem 1.1 in Section 5, we begin with some identities, the first two of which in the next result follow from (1.7) and (1.18) while the rest appear in [10, Eqs. (2.2f), (2.4a), (2.4b), (2.4c) and (2.4f)]. Moreover, we frequently use without mention

$$j(q;q^2)=\frac{J_1^2}{J_2},\quad j(q;q^3)=J_1,\quad j(q;q^6)=\frac{J_1J_6^2}{J_2J_3},$$
$$j(-1;q)=2\frac{J_2^2}{J_1},\quad j(-q;q^3)=\frac{J_2J_3^2}{J_1J_6}\quad\text{and}\quad j(-q;q^6)=\frac{J_2^2J_3J_{12}}{J_1J_4J_6}.$$

Proposition 2.6. *For generic $x,y,z\in\mathbb{C}^*$, we have*

$$j(xq^{-1};q^2)j(x^{-1}q^2;q^2)=x^2q^{-1}j(x^{-1};q)\frac{J_2^2}{J_1}, \tag{2.20}$$

$$\frac{j(-x;q)}{j(-x^2q;q^2)j(x;q)}=-\frac{j(-x^{-1};q)}{j(-x^{-2}q;q^2)j(x^{-1};q)}, \tag{2.21}$$

$$j(z;q)=\sum_{k=0}^{n-1}(-1)^kq^{k(k-1)/2}z^kj((-1)^{n+1}q^{n(n-1)/2+nk}z^n;q^{n^2}), \tag{2.22}$$

$$j(qx^3;q^3)+xj(q^2x^3;q^3)=J_1\frac{j(x^2;q)}{j(x;q)}, \tag{2.23}$$

$$j(x;q)j(y,q)=j(-xy;q^2)j(-qx^{-1}y;q^2)-xj(-xyq;q^2)j(-x^{-1}y;q^2), \tag{2.24}$$

$$\frac{j(y;q)}{j(-y;q)}-\frac{j(x;q)}{j(-x;q)}=2x\frac{j(y/x;q^2)j(qxy;q^2)}{j(-x;q)j(-y;q)} \tag{2.25}$$

and

$$\frac{j(zx;q)}{j(x;q)}=\frac{J_n^3j(z;q)}{J_1^3j(x^n;q^n)}\sum_{k=0}^{n-1}x^k\frac{j(zx^nq^k;q^n)}{j(zq^k;q^n)}. \tag{2.26}$$

Next, we state results which can be found in [1, Eqs. (1.6) and (1.7)] or deduced from (1.7), (2.22) and (2.23).

Proposition 2.7. *If w is a primitive third root of unity and $x \in \mathbb{C}^*$ is generic, then*

$$j(w;q) = (1-w)J_3, \tag{2.27}$$

$$j(-w;q) = (1+w)\frac{J_1^2 J_6}{J_2 J_3}, \tag{2.28}$$

$$j(-wq;q^2) = \frac{J_1 J_4 J_6^2}{J_2 J_3 J_{12}}, \tag{2.29}$$

$$j(-wq;q^3) = J_9\left(\frac{j(q^2;q^9)}{j(-q;q^9)} - w^2 q\frac{j(q^8;q^9)}{j(-q^4;q^9)}\right), \tag{2.30}$$

$$j(-wq;q^6) = J_{18}\left(\frac{j(q^{10};q^{18})}{j(-q^5;q^{18})} + wq\frac{j(q^{14};q^{18})}{j(-q^7;q^{18})}\right) \tag{2.31}$$

and

$$j(x;q)j(xw;q)j(xw^2;q) = \frac{J_1^3}{J_3}j(x^3;q^3). \tag{2.32}$$

Before proceeding, recall that given a q-series $F(q)$ with integral powers, its N-dissection is a representation of the form

$$F(q) = \sum_{k=0}^{N-1} q^k F_k(q^k)$$

where $F_k(q)$ is a series in q with integral powers. The study of N-dissections ostensibly began in Ramanujan's lost notebook [19] and has now become its own extensive literature. As it is not possible to give a comprehensive overview of this area here, we only mention the classical result of Atkin and Swinnerton-Dyer on 5 and 7 dissections of the rank of partitions [2] and the recent impressive work [5]. The final collection of results concerns 3-dissections of certain eta quotients. The following two identities are [3, Eq. (3.9)] and the $a=-q$, $b=-q^5$, $n=3$ case of [4, Entry 31, page 48], respectively.

Proposition 2.8. *We have*

$$\frac{1}{J_1^3} = W_0(q^3) + qW_1(q^3) + q^2 W_2(q^3) \tag{2.33}$$

where

$$W_0(q) := \frac{J_3^9}{J_1^{12}}\left(\frac{1}{w^2(q)} + 8qw(q) + 16q^2w^4(q)\right),$$

$$W_1(q) := \frac{J_3^9}{J_1^{12}}\left(\frac{3}{w(q)} + 12qw^2(q)\right), \qquad W_2(q) := 9\frac{J_3^9}{J_1^{12}}$$

and

$$w(q) := \frac{J_1 J_6^3}{J_2 J_3^3}.$$

Proposition 2.9. *We have*

$$\frac{J_1J_6}{J_2J_3^2} = f_0(q^3) + qf_1(q^3) + q^2f_2(q^3) \tag{2.34}$$

where

$$f_0(q) := \frac{j(q^7;q^{18})}{J_1J_2}, \qquad f_1(q) := -\frac{j(q^5;q^{18})}{J_1J_2} \qquad \text{and} \qquad f_2(q) := -q\frac{j(q;q^{18})}{J_1J_2}.$$

The following 3-dissections appear to be new. As the proofs are comparable, we only give details once.

Proposition 2.10. *We have*

$$\frac{J_2^4J_8}{J_1J_4^3} = g_0(q^3) + qg_1(q^3) + q^2g_2(q^3) \tag{2.35}$$

where

$$g_0(q) := \frac{J_1J_2^2J_8^2J_{12}^2}{J_4^5J_{24}}, \qquad g_1(q) := \frac{J_2^7J_3J_8^2J_{12}^4}{J_1^2J_4^7J_6^3J_{24}} \qquad \text{and} \qquad g_2(q) = -2\frac{J_2^2J_6^2J_8^3}{J_3J_4^5}.$$

Proof. If we multiply both sides of (2.35) by $q^{12}\frac{J_1J_4^3J_{12}^6J_{18}^8J_{36}^2}{J_2^4J_8}$, then the claim is equivalent to

$$\begin{aligned} \eta^6(q^{12})\eta^8(q^{18})\eta^2(q^{36}) = & \frac{\eta(q)\eta(q^3)\eta^3(q^4)\eta^2(q^6)\eta(q^{12})\eta^8(q^{18})\eta^2(q^{24})\eta^4(q^{36})}{\eta^4(q^2)\eta(q^8)\eta(q^{72})} \\ & + \frac{\eta(q)\eta^3(q^4)\eta^7(q^6)\eta(q^9)\eta^5(q^{18})\eta^2(q^{24})\eta^6(q^{36})}{\eta^4(q^2)\eta^2(q^3)\eta(q^8)\eta(q^{12})\eta(q^{72})} \\ & - 2\frac{\eta(q)\eta^3(q^4)\eta^2(q^6)\eta(q^{12})\eta^{10}(q^{18})\eta^3(q^{24})\eta^2(q^{36})}{\eta^4(q^2)\eta(q^8)\eta(q^9)} \end{aligned} \tag{2.36}$$

where $\eta(q) := q^{1/24}J_1$. Using [18, Theorems 1.64 and 1.65], we see that both sides of (2.36) are holomorphic modular forms of level 72 and weight 8. Since the corresponding space of modular forms has dimension 92 [18, Theorem 1.34], it suffices to check that the first 92 coefficients in the q-series expansion of (2.36) agree. This completes the proof of (2.35). □

Proposition 2.11. *We have*

$$\frac{J_2^3}{J_1J_8} = h_0(q^3) + qh_1(q^3) + q^2h_2(q^3) \tag{2.37}$$

where

$$h_0(q) := \frac{J_4^4J_6^2}{J_2J_3J_8^3}, \qquad h_1(q) := \frac{J_1J_4J_6J_{24}}{J_8^2J_{12}} \qquad \text{and} \qquad h_2(q) := -\frac{J_2^5J_3J_{12}J_{24}}{J_1^2J_4J_6^2J_8^2}.$$

Proposition 2.12. *We have*

$$\frac{J_2}{J_4^2} = I_0(q^3) + qI_1(q^3) + q^2I_2(q^3) \tag{2.38}$$

where

$$I_0(q) := \frac{J_2^2J_6^3}{J_4^6}, \qquad I_1(q) := q\frac{J_2^4J_{12}^6}{J_4^8J_6^3} \qquad \text{and} \qquad I_2(q) := -\frac{J_2^3J_{12}^3}{J_4^7}.$$

3. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. We follow the strategy in [15]. Consider the left-hand side of (2.9) and take $z' = -1$ in (2.10). This yields

$$\begin{aligned}\frac{1}{M}\sum_{j=1}^{M-1}\zeta_M^{-aj}\overline{S}_d(\zeta_M^j;q) &= \frac{1}{M}\sum_{j=1}^{M-1}\zeta_M^{-aj}(1-\zeta_M^j)(1-2m(\zeta_M^{-2j}q^{d^2},q^{2d^2},-1)+2\Lambda(d,\zeta_M^j,z_0,-1))\\ &= \frac{1}{M}\left(\sum_{j=0}^{M-1}\zeta_M^{-aj}-\sum_{j=0}^{M-1}\zeta_M^{-(a-1)j}\right)-\frac{2}{M}\left(\sum_{j=0}^{M-1}\zeta_M^{-aj}m(\zeta_M^{-2j}q^{d^2},q^{2d^2},-1)\right.\\ &\left.-\sum_{j=0}^{M-1}\zeta_M^{-(a-1)j}m(\zeta_M^{-2j}q^{d^2},q^{2d^2},-1)\right)\\ &+\frac{2}{M}\sum_{j=1}^{M}\zeta_{\frac{M}{2}}^{-\frac{aj}{2}}(1-\zeta_M^j)\Lambda(d,\zeta_M^j,z_0,-1). \qquad (3.1)\end{aligned}$$

To prove (1.11), we first use (2.7) to observe that the first two sums in the second line of (3.1) equal 0 if $a < M$ and 1 if $a = M$. For the third sum, we split it into two further sums. We then reindex the resulting second sum by $j \to \frac{M}{2} + j$, use that $\zeta_M^{-2} = \zeta_{\frac{M}{2}}^{-1}$, write $a = 2t$ where $t > 0$, apply (2.3) and take $k = t-1$, $n = \frac{M}{2}$, $z = -1$, $x = q^{-d^2}$ and $q \to q^{2d^2}$ in (2.2) to obtain

$$\begin{aligned}\sum_{j=0}^{M-1}\zeta_M^{-aj}m(\zeta_M^{-2j}q^{d^2},q^{2d^2},-1) &= 2\sum_{j=0}^{\frac{M}{2}-1}\zeta_M^{-aj}m(\zeta_{\frac{M}{2}}^{-j}q^{d^2},q^{2d^2},-1)\\ &= M(-1)^{\frac{a}{2}-1}q^{-\frac{a^2}{4}+\frac{a}{2}(1-d^2)}m((-1)^{\frac{M}{2}+1}q^{\frac{M^2}{4}-\frac{aM}{2}+\frac{M}{2}(1-d^2)},q^{\frac{M^2}{2}},z')\\ &\quad + q^{-d^2}M\Psi_{\frac{a}{2}-1}^{\frac{M}{2}}(q^{-d^2},-1,z';q^2).\end{aligned}$$

For the fourth sum, we similarly split it into two further sums, then reindex the resulting second sum by $j \to \frac{M}{2} + j$. We then use that $\zeta_M^{\frac{M}{2}} = -1$ to obtain 0. In total, this yields (1.11).

To prove (1.12), we first use (2.7) to note that the first two sums in the second line of (3.1) equal 0. For the third sum, we first use that if ζ_M is a primitive M-th root of unity, then so is ζ_M^2 as M is odd and then take $k = \frac{2M-a}{2}$, $n = M$, $z = -1$, $x = q^{d^2}$ and $q \to q^{2d^2}$ in (2.2) to obtain

$$\begin{aligned}\sum_{j=0}^{M-1}\zeta_M^{-aj}m(\zeta_M^{-2j}q^{d^2},q^{2d^2},-1) &= Mq^{-d^2(\frac{2M-a}{2})^2}(-1)^{\frac{2M-a}{2}}m(q^{d^2M(a-M)},q^{2d^2M^2},z')\\ &\quad + M\Psi_{\frac{2M-a}{2}}^{M}(q^{d^2},-1,z';q^{2d^2}).\end{aligned}$$

For the fourth sum, we take $k = \frac{M+1-a}{2}$, $n = M$, $z = -1$, $x = q^{d^2}$ and $q \to q^{2d^2}$ in (2.2) to obtain

$$\sum_{j=0}^{M-1} \zeta_M^{-(a-1)j} m(\zeta_M^{-2j} q^{d^2}, q^{2d^2}, -1) = M q^{-d^2(\frac{a+M-1}{2})^2} (-1)^{\frac{a+M-1}{2}} m(q^{d^2(M-aM)}, q^{2d^2M^2}, z'') + M\Psi^M_{\frac{a+M-1}{2}}(q^{d^2}, -1, z''; q^{2d^2}).$$

In total, this yields (1.12).

Finally, to prove (1.13), we first use (2.7) to see that the first two sums in the second line of (3.1) equal 0 unless $a = M$, in which case the sum is M. For the third sum, we take $k = \frac{M-a}{2}$, $n = M$, $z = -1$, $x = q^{d^2}$ and $q \to q^{2d^2}$ in (2.2) to obtain

$$\sum_{j=0}^{M-1} \zeta_M^{-aj} m(\zeta_M^{-2j} q^{d^2}, q^{2d^2}, -1) = M q^{-d^2(\frac{M-a}{2})^2} (-1)^{\frac{M-a}{2}} m(q^{d^2aM}, q^{2d^2M^2}, z') + M\Psi^M_{\frac{M-a}{2}}(q^{d^2}, -1, z'; q^{2d^2}).$$

For the fourth sum, we take $k = \frac{2M-a+1}{2}$, $n = M$, $z = -1$, $x = q^{d^2}$ and $q \to q^{2d^2}$ in (2.2) to obtain

$$\sum_{j=0}^{M-1} \zeta_M^{-(a-1)j} m(\zeta_M^{-2j} q^{d^2}, q^{2d^2}, -1) = M q^{-d^2(\frac{2M-a+1}{2})^2} (-1)^{\frac{2M-a+1}{2}} m(q^{d^2M(a-M-1)}, q^{2d^2M^2}, z'') + M\Psi^M_{\frac{2M-a+1}{2}}(q^2, -1, z''; q^{2d^2}).$$

In total, this yields (1.13). □

Proof of Theorem 1.2. Consider the left-hand side of (2.9) and take $z' = -1$ in (2.11). This yields

$$\begin{aligned}
\frac{1}{M}\sum_{j=1}^{M-1} \zeta_M^{-aj} \overline{S}_d(\zeta_M^j; q) &= \frac{1}{M}\sum_{j=1}^{M-1} \zeta_M^{-aj}(1-\zeta_M^j)\Bigg(-1 + 2m((-1)^{\frac{d}{2}+1}\zeta_M^j q^{\frac{d^2}{4}}, q^{\frac{d^2}{2}}, -1) \\
&\qquad + 2(-1)^{\frac{d}{2}}\zeta_M^j q^{-\frac{d^2}{4}} \Psi_0^{\frac{d}{2}}(\zeta_M^{\frac{2j}{d}} q^{1-d}, q, -1; q^2)\Bigg) \\
&= -\frac{1}{M}\left(\sum_{j=0}^{M-1}\zeta_M^{-aj} - \sum_{j=0}^{M-1}\zeta_M^{-(a-1)j}\right) \\
&+ \frac{2}{M}\Bigg(\sum_{j=0}^{M-1}\zeta_M^{-aj} m((-1)^{\frac{d}{2}+1}\zeta_M^j q^{\frac{d^2}{4}}, q^{\frac{d^2}{2}}, -1) \\
&- \sum_{j=0}^{M-1}\zeta_M^{-(a-1)j} m((-1)^{\frac{d}{2}+1}\zeta_M^j q^{\frac{d^2}{4}}, q^{\frac{d^2}{2}}, -1)\Bigg) \\
&+ \frac{2}{M}(-1)^{\frac{d}{2}} q^{-\frac{d^2}{4}} \sum_{j=1}^{M-1} \zeta_M^{j-aj}(1-\zeta_M^j)\Psi_0^{\frac{d}{2}}(\zeta_M^{\frac{2j}{d}} q^{1-d}, q, -1; q^2).
\end{aligned} \tag{3.2}$$

To prove (1.14), we first use (2.7) to observe that the first two sums in the second line of (3.2) equal 0 unless $a = 1$, in which case we obtain 1. For the third sum, we take $k = a$, $n = M$, $z = -1$, $x = (-1)^{\frac{d}{2}+1}q^{\frac{d^2}{4}}$ and $q \to q^{\frac{d^2}{2}}$ in (2.2) to obtain

$$\sum_{j=0}^{M-1} \zeta_M^{-aj} m(\zeta_M^j(-1)^{\frac{d}{2}+1}q^{\frac{d^2}{4}}, q^{\frac{d^2}{2}}, -1) = M(-1)^{\frac{da}{2}} q^{-\frac{d^2a^2}{4}} m((-1)^{1+\frac{dM}{2}} q^{\frac{d^2}{4}(M^2-2Ma)}, q^{\frac{d^2M^2}{2}}, z') + M\Psi_a^M((-1)^{\frac{d}{2}+1}q^{\frac{d^2}{4}}, -1, z'; q^{\frac{d^2}{2}}).$$

For the fourth sum, we take $k = a-1$, $n = M$, $z = -1$, $x = (-1)^{\frac{d}{2}+1}q^{\frac{d^2}{4}}$ and $q \to q^{\frac{d^2}{2}}$ in (2.2) to obtain

$$\sum_{j=0}^{M-1} \zeta_M^{-(a-1)j} m((-1)^{\frac{d}{2}+1}\zeta_M^j q^{\frac{d^2}{4}}, q^{\frac{d^2}{2}}, -1)$$
$$= M(-1)^{\frac{d}{2}(a-1)+1} q^{-\frac{d^2}{4}(a^2-2a+1)} m((-1)^{1+\frac{dM}{2}} q^{\frac{d^2}{4}(M^2-2M(a-1))}, q^{\frac{d^2M^2}{2}}, z'') + M\Psi_{a-1}^M((-1)^{\frac{d}{2}+1}q^{\frac{d^2}{4}}, -1, z''; q^{\frac{d^2}{2}}).$$

In total, this yields (1.14). □

4. Single generalized rank deviations

In this section, we briefly discuss why there is no loss in generality in considering pairs of generalized rank deviations in Theorems 1.1 and 1.2. For M odd, we have

$$\overline{D}_d\Big(\frac{M+1}{2}, M\Big) + \overline{D}_d\Big(\frac{M-1}{2}, M\Big) = 2\overline{D}_d\Big(\frac{M-1}{2}, M\Big)$$

and so Theorem 1.1 can be used to find a formula for any single $\overline{D}_d(a, M)$. Precisely, we prove the following result.

Proposition 4.1. *Let M be odd. For $0 \le n \le \frac{M+1}{2}$, we have*

$$\overline{D}_d\Big(\frac{M+1}{2}+n, M\Big) = \frac{1}{2}\Bigg(\sum_{i=0}^{n}\Big[\overline{D}_d\Big(\frac{M+1}{2}-n+2i, M\Big) + \overline{D}_d\Big(\frac{M+1}{2}-n+2i-1, M\Big)\Big] - \sum_{i=0}^{n-1}\Big[\overline{D}_d\Big(\frac{M+1}{2}-n+2i+1, M\Big) + \overline{D}_d\Big(\frac{M+1}{2}-n+2i, M\Big)\Big]\Bigg).$$

Proof. We begin by letting

$$\overline{D}_n := \sum_{i=0}^{n}\Big[\overline{D}_d\Big(\frac{M+1}{2}+i, M\Big) + \overline{D}_d\Big(\frac{M-1}{2}-i, M\Big)\Big].$$

By (1.15), we have

$$\overline{D}_n = 2\sum_{i=0}^{n}\overline{D}_d\Big(\frac{M+1}{2}+i, M\Big)$$

and so

$$\overline{D}_d\Big(\frac{M+1}{2}+n, M\Big) = \frac{\overline{D}_n - \overline{D}_{n-1}}{2}. \tag{4.1}$$

After reindexing, one obtains

$$\overline{D}_n = \sum_{i=0}^{n}\left[\overline{D}_d\Big(\frac{M+1}{2}-n+2i, M\Big) + \overline{D}_d\Big(\frac{M+1}{2}-n+2i-1, M\Big)\right]. \tag{4.2}$$

Combining (4.1) and (4.2) yields the result. □

For M even, one can use the following result.

Proposition 4.2. *Let M be even, $0 \le n \le M-1$ and z_0, $z' \in \mathbb{C}^*$ be generic. For d odd, we have*

$$\overline{D}_d(n, M) = \frac{(-1)^n}{M}\mathcal{O}_d(-1;q) + \frac{1}{M}\sum_{k=1}^{\frac{M}{2}-1}\left(\frac{1-\zeta_M^k}{1+\zeta_M^k}\right)\left(\zeta_M^{-kn}+\zeta_M^{kn}\right) \times \left(1 - 2m(\zeta_M^{-2k}q^{d^2}, q^{2d^2}, z') + 2\Lambda(d, \zeta_M^k, z_0, z')\right). \tag{4.3}$$

For d even, we have

$$\overline{D}_d(n, M) = \frac{(-1)^n}{M}\mathcal{O}_d(-1;q) + \frac{1}{M}\sum_{k=1}^{\frac{M}{2}-1}\left(\frac{1-\zeta_M^k}{1+\zeta_M^k}\right)\left(\zeta_M^{-kn}+\zeta_M^{kn}\right)\Bigg(-1 + 2m((-1)^{\frac{d}{2}+1}\zeta_M^k q^{\frac{d^2}{4}}, q^{\frac{d^2}{2}}, z') + 2(-1)^{\frac{d}{2}}\zeta_M^k q^{-\frac{d^2}{4}}\Psi_0^{\frac{d}{2}}(\zeta_M^{\frac{2k}{d}}q^{1-d}, q, z'; q^2)\Bigg). \tag{4.4}$$

Proof. From (1.3) and (2.7), one deduces

$$\overline{D}_d(n, M) = \frac{1}{M}\sum_{k=1}^{M-1}\mathcal{O}_d(\zeta_M^k;q)\zeta_M^{-kn} \tag{4.5}$$

for all M. For M even, we remove the $k = \frac{M}{2}$ term from (4.5), split it into two further sums, then reindex the resulting second sum by $k \to M-k$. This yields

$$\overline{D}_d(n, M) = \frac{(-1)^n}{M}\mathcal{O}_d(-1;q) + \frac{1}{M}\sum_{k=1}^{\frac{M}{2}-1}\left(\zeta_M^{-kn}+\zeta_M^{kn}\right)\mathcal{O}_d(\zeta_M^k;q). \tag{4.6}$$

Finally, (4.3) and (4.4) follow from applying (2.8), (2.10) and (2.11) to (4.6). □

5. An application of Theorem 1.1

As an application of Theorem 1.1, we compute the 3-dissection of $\mathcal{O}_3(\zeta_3;q)$ (cf. [12, Theorem 1.3]). In principle, one can reduce the number of eta quotients appearing in our result. We do not pursue this further. Let

$$\begin{gathered}A := j(-q^{12};q^{27}), \quad B := qj(-q^{21};q^{27}), \quad C := q^2 j(-q^3;q^{27}), \quad D := \frac{j(q^{60};q^{108})}{j(-q^{30};q^{108})}, \\ E := q^6\frac{j(q^{84};q^{108})}{j(-q^{42};q^{108})}, \quad F := \frac{j(q^{24};q^{108})}{j(-q^{12};q^{108})}, \quad G := q^{12}\frac{j(q^{96};q^{108})}{j(-q^{48};q^{108})},\end{gathered} \tag{5.1}$$

$$\mathcal{G}_N := \sum_{\substack{k,\,l,\,m\,\in\,\{0,1,2\} \\ k+l+m\,\equiv\,N \pmod 3}} q^{k+l+m} g_k(q^3) W_l(q^3) f_m(q^3) \tag{5.2}$$

and

$$\mathcal{H}_N := \sum_{\substack{k,\,l,\,m\,\in\,\{0,1,2\} \\ k+l+m\,\equiv\,N \pmod 3}} q^{k+l+m} h_k(q^3) W_l(q^3) f_m(q^3) \tag{5.3}$$

where $N = 0$, 1 or 2 and the functions $W_i(q)$, $f_i(q)$, $g_i(q)$ and $h_i(q)$ are given in (2.33)–(2.37). Also, recall the functions $I_i(q)$ in (2.38).

Theorem 5.1. *We have*

$$\mathcal{O}_3(\zeta_3;q) = \overline{\mathcal{B}}_0(q^3) + q\mathcal{B}_1(q^3) + q^2\mathcal{B}_2(q^3) \tag{5.4}$$

where

$$\begin{aligned}\overline{\mathcal{B}}_0(q^3) &:= 6q^{-36} m(q^{-27}, q^{162}, -1) \\ &\quad - \frac{3}{2}q^{-9}\frac{J_{18}J_{27}J_{108}J_{162}^5}{J_{36}^2 J_{54} J_{81} J_{324}^3}\left(\frac{j(q^{27};q^{162})}{j(-q^{27};q^{162})} + \frac{j(q^{81};q^{162})}{j(-q^{81};q^{162})}\right) + \mathcal{B}_0(q^3)\end{aligned}$$

and

$$\begin{aligned}\mathcal{B}_N(q^3) &:= 3q^3\frac{J_6^3 J_9 J_{108}}{J_3 J_{18} J_{36}} q^N I_N(q^3) \\ &\quad + \frac{J_3^2 J_6^2 J_{36}}{J_{12} J_{18}^2}\Bigg(\frac{J_3^3 J_{12}^2 J_{18}^2 J_{72} J_{108}^2}{J_6^4 J_9 J_{24} J_{36} J_{54} J_{216}} \sum_{\substack{k,\,l\,\in\,\{0,1,2\} \\ k+l\,\equiv\,N \pmod 3}} q^{k+l} g_k(q^3) W_l(q^3) \\ &\quad - 2q^2\frac{J_{12}^2 J_{108}}{J_6 J_{24}}\Big(\big(2AD - AE\big)\mathcal{G}_{N+1} - \big(BD + BE\big)\mathcal{G}_N + \big(2CE - CD\big)\mathcal{G}_{2+N}\Big) \\ &\quad + q^5\frac{J_3^3 J_{18} J_{24} J_{36}^2 J_{216}}{J_6^3 J_9 J_{12} J_{72} J_{108}} \sum_{\substack{k,\,l\,\in\,\{0,1,2\} \\ k+l\,\equiv\,N+1 \pmod 3}} q^{k+l} h_k(q^3) W_l(q^3) \\ &\quad - 2q\frac{J_{24}J_{108}}{J_{12}}\Big(\big(2AG + AF\big)\mathcal{H}_{N+2} - \big(2BF + BG\big)\mathcal{H}_{N+1} - \big(CG - CF\big)\mathcal{H}_N\Big)\Bigg)\end{aligned}$$

for $N = 0$, 1 *and* 2.

Proof of Theorem 5.1. We begin by decomposing $\mathcal{O}_3(\zeta_3;q)$ as follows. By (1.3)–(1.5) and (1.15), we have

$$\begin{aligned}\mathcal{O}_3(\zeta_3;q) &= \sum_{n\geq 0}\sum_{s=0}^{2}\overline{N}_3(s,3,n)\zeta_3^s q^n\\ &= \sum_{n\geq 0}\left(\overline{N}_3(0,3,n)+\zeta_3\overline{N}_3(1,3,n)+\zeta_3^2\overline{N}_3(2,3,n)\right)q^n\\ &= \sum_{n\geq 0}\left(\overline{N}_3(0,3,n)-\overline{N}_3(2,3,n)\right)q^n\\ &= \overline{D}_3(3,3)+\overline{D}_3(2,3)-\left(\overline{D}_3(2,3)+\overline{D}_3(1,3)\right)\end{aligned} \tag{5.5}$$

where we have used that $\overline{N}_3(1,3,n)=\overline{N}_3(2,3,n)$ and $1+\zeta_3+\zeta_3^2=0$. We now take $d=3$, $a=M=3$, $z'=z''=z_0=-1$ in (iii) and $d=3$, $a=2$, $M=3$, $z'=z''=z_0=-1$ in (ii) of Theorem 1.1, respectively, and simplify using (2.3), (2.5) and the fact that

$$\Psi_0^3(q^9,-1,-1;q^{18})=0$$

to obtain from (5.5)

$$\begin{aligned}\mathcal{O}_3(\zeta_3;q) = 6q^{-36}m(q^{-27},q^{162},-1) - 2\zeta_3\Lambda(3,\zeta_3,-1,-1) - 2\zeta_3^2\Lambda(3,\zeta_3^2,-1,-1)\\ + 4\Psi_2^3(q^9,-1,-1;q^{18}) - 2\Psi_1^3(q^9,-1,-1;q^{18}).\end{aligned} \tag{5.6}$$

By (1.10), (1.17), (1.18) and (2.20), we have

$$\begin{aligned}-2\zeta_3\Lambda(3,\zeta_3,-1,-1) &= -2q^{-1}\zeta_3^{5/3}\Psi_1^3(\zeta_3^{-2/3}q^3,-1,-1;q^2)\\ &\quad+\frac{2\zeta_3^{5/3}J_2^3}{3qj(-1;q^2)}\sum_{t=0}^{2}\zeta_3^{-t}\frac{j(-\zeta_3^{t+1/3}/q;q^2)j(-\zeta_3^{-t-1/3}q^2;q^2)}{j(\zeta_3^{t+1/3}/q;q^2)j(\zeta_3^{-t-1/3}q^2;q^2)j(-\zeta_3^{-2t-2/3}q^3;q^2)}\\ &= -\frac{\zeta_3 J_2J_6J_{18}^4}{2J_4^2J_{36}^2 j(-\zeta_3q^9;q^{18})}\left(\frac{j(\zeta_3q^{15};q^{18})}{j(-\zeta_3q^{15};q^{18})}+\frac{j(\zeta_3q^{21};q^{18})}{j(-\zeta_3q^{21};q^{18})}\right)\\ &\quad+\frac{\zeta_3J_2^4}{3J_4^2}\sum_{t=0}^{2}\frac{j(-\zeta_3^{t+1/3}q;q)}{j(\zeta_3^{t+1/3}q;q)j(-\zeta_3^{2t+2/3}q;q^2)}.\end{aligned} \tag{5.7}$$

Similarly,

$$\begin{aligned}-2\zeta_3^2\Lambda(3,\zeta_3^2,-1,-1) &= \frac{\zeta_3^2J_2J_6J_{18}^4}{2J_4^2J_{36}^2j(-\zeta_3^2q^9;q^{18})}\left(\frac{j(\zeta_3q^{15};q^{18})}{j(-\zeta_3q^{15};q^{18})}+\frac{j(\zeta_3q^{21};q^{18})}{j(-\zeta_3q^{21};q^{18})}\right)\\ &\quad-\frac{\zeta_3^2J_2^4}{3J_4^2}\sum_{t=0}^{2}\frac{j(-\zeta_3^{t+2/3};q)}{j(\zeta_3^{t+2/3};q)j(-\zeta^{2t+4/3}q;q^2)}\end{aligned} \tag{5.8}$$

after applying (1.17), (1.18) and (2.21). Using (1.17), (1.18), the $x=\zeta_3^2q^{15}$ and $y=\zeta_3q^{15}$ case of (2.25), (2.28) and (2.32), one confirms

$$\frac{j(\zeta_3q^{15};q^{18})}{j(-\zeta_3q^{15};q^{18})}+\frac{j(\zeta_3q^{21};q^{18})}{j(-\zeta_3q^{21};q^{18})} = -2q^3\zeta_3^2(1-\zeta_3^2)\frac{J_6^2J_9^2J_{36}^2J_{54}^2}{J_3J_{18}^6J_{27}}. \tag{5.9}$$

Hence, taking the sum of the first terms on the right-hand sides of (5.7) and (5.8), respectively, combined with (2.29) and (5.9) yields after simplification

$$-(\zeta_3-\zeta_3^2)\frac{J_2J_6J_{18}^4}{2J_4^2J_{36}^2j(-\zeta_3q^9;q^{18})}\left(\frac{j(\zeta_3q^{15};q^{18})}{j(-\zeta_3q^{15};q^{18})}+\frac{j(\zeta_3q^{21};q^{18})}{j(-\zeta_3q^{21};q^{18})}\right)=3q^3\frac{J_2J_6^3J_9J_{108}}{J_3J_4^2J_{18}J_{36}}. \tag{5.10}$$

It now remains to consider the sum of the second terms on the right-hand sides of (5.7) and (5.8), respectively. This is

$$(\zeta_3-\zeta_3^2)\frac{J_2^4}{3J_4^2}\sum_{t=0}^{2}\frac{j(-\zeta_3^{t+1/3}q;q)}{j(\zeta_3^{t+1/3}q;q)j(-\zeta_3^{2t+2/3}q;q^2)}. \tag{5.11}$$

We now apply (1.18) to the summand of (5.11), then take $x=\zeta_3^{t+1/3}$, $z=-1$ and $n=3$ in (2.26) to obtain

$$-(\zeta_3-\zeta_3^2)\frac{2J_2^6J_3^3}{3J_1^4J_4^2j(\zeta_3;q^3)}\sum_{k=0}^{2}\frac{j(-\zeta_3q^k;q^3)}{j(-q^k;q^3)}\sum_{t=0}^{2}\frac{\zeta_3^{(t+1/3)k}}{j(-\zeta_3^{2(t+1/3)}q;q^2)}. \tag{5.12}$$

By (2.32),

$$j(-\zeta_3^{2/3}q;q^2)j(-\zeta_3^{8/3}q;q^2)j(-\zeta_3^{14/3}q;q^2)=\frac{J_2^3J_3J_{12}J_{18}^2}{J_6^2J_9J_{36}} \tag{5.13}$$

and so using (2.27) and (5.13) turns (5.12) into

$$-\zeta_3\frac{2J_2^3J_3^2J_6^2J_{36}}{3J_1^4J_4^2J_{12}J_{18}^2}\sum_{k=0}^{2}\frac{j(-\zeta_3q^k;q^3)}{j(-q^k;q^3)}\sum_{t=0}^{2}\zeta_3^{(t+1/3)k}j(-\zeta_3^{2(t+4/3)}q;q^2)j(-\zeta_3^{2(t+7/3)}q;q^2). \tag{5.14}$$

Note that

$$\begin{aligned}&j(-\zeta_3^{2t+8/3}q;q^2)j(-\zeta_3^{2t+5/3}q;q^2)\\&\qquad=\frac{J_2J_8J_{12}^2}{J_4J_6J_{24}}j(-\zeta_3^{4(t+1/3)}q^2;q^4)-\zeta_3^{2t+2/3}q\frac{J_4^2J_{24}}{J_8J_{12}}j(-\zeta_3^{4(t+1/3)}q^4;q^4)\end{aligned}$$

by (2.24), (2.27) and (2.29) and so (5.14) equals

$$\begin{aligned}-\frac{2J_2^3J_3^2J_6^2J_{36}}{3J_1^4J_4^2J_{12}J_{18}^2}\Bigg(&\zeta_3\frac{J_2J_8J_{12}^2}{J_4J_6J_{24}}\sum_{k=0}^{2}\frac{j(-\zeta_3q^k;q^3)}{j(-q^k;q^3)}\sum_{t=0}^{2}\zeta_3^{(t+1/3)k}j(-\zeta_3^{4(t+1/3)}q^2;q^4)\\&-q\frac{J_4^2J_{24}}{J_8J_{12}}\sum_{k=0}^{2}\zeta_3^{(k+5)/3}\frac{j(-\zeta_3q^k;q^3)}{j(-q^k;q^3)}\sum_{t=0}^{2}\zeta_3^{(k+2)t}j(-\zeta_3^{(t+4/3)}q^4;q^4)\Bigg).\end{aligned} \tag{5.15}$$

We now simplify the first line in (5.15). By (1.7) and (2.7),

$$\begin{aligned}\sum_{t=0}^{2}\zeta_3^{(t+1/3)k}j(-\zeta_3^{4(t+1/3)}q^2;q^4)&=\sum_{n\in\mathbb{Z}}q^{2n^2}\sum_{t=0}^{2}\zeta_3^{(t+1/3)(4n+k)}\\&=3\sum_{n\equiv 2k\ (\mathrm{mod}\ 3)}q^{2n^2}\zeta_3^{(4n+k)/3}\\&=3\sum_{s\in\mathbb{Z}}q^{36s(s-1)/2+(24k+18)s+8k^2}\zeta_3^{s}\\&=3q^{8k^2}j(-\zeta_3q^{24k+18};q^{36}).\end{aligned} \tag{5.16}$$

First, applying (2.29) to (5.16), the $k=0$ term in the first line of (5.15) equals

$$-\frac{3J_3^3J_{18}^2J_{72}J_{108}^2}{2J_6^3J_9J_{36}J_{54}J_{216}}. \tag{5.17}$$

Next, applying (2.31) to (5.16) and the $z=-\zeta_3 q$, $q\to q^3$ and $n=3$ case of (2.22) and simplifying, the $k=1$ term in the first line of (5.15) equals

$$3q^2\frac{J_1J_6J_{108}}{J_2J_3^2}\left(AD-BE-CD+CE+\zeta_3(AE+BD-BE-CD)\right) \tag{5.18}$$

where A, B, C, D and E are given in (5.1). Finally, applying (1.17), (1.18) and (2.31) to (5.16) and simplifying, the $k=2$ term in the first line of (5.15) equals

$$3q^2\frac{J_1J_6J_{108}}{J_2J_3^2}(AD-AE-BD+CE+\zeta_3(-AE-BD+BE-CD)). \tag{5.19}$$

Combining (5.17)–(5.19) and simplifying, the first line of (5.15) equals

$$-\frac{J_2^4J_3^2J_6J_8J_{12}J_{36}}{J_1^4J_4^3J_{18}^2J_{24}}\left(-\frac{J_3^3J_{18}^2J_{72}J_{108}^2}{J_6^3J_9J_{36}J_{54}J_{216}}+2q^2\frac{J_1J_6J_{108}}{J_2J_3^2}(2AD-AE-BD-BE-CD+2CE)\right). \tag{5.20}$$

We now simplify the second line in (5.15). Similar to (5.16), we find

$$\sum_{t=0}^{2}\zeta_3^{(k+2)t}j(-\zeta_3^{t+4/3};q^4)=3\zeta_3^{(5k-5)/3}q^{2k^2-6k+4}j(-\zeta_3q^{36-12k};q^{36}). \tag{5.21}$$

Applying (2.28) to (5.21), the $k=0$ term in the second line of (5.15) equals

$$3q^4\frac{J_3^3J_{18}J_{36}^2J_{216}}{2J_6^3J_9J_{72}J_{108}}. \tag{5.22}$$

Next, applying (1.18) and (2.30) to (5.21) and (again) the $z=-\zeta_3q$, $q\to q^3$ and $n=3$ case of (2.22) and simplifying, the $k=1$ term in the second line of (5.15) becomes

$$3\frac{J_1J_6J_{108}}{J_2J_3^2}(-AF-AG+BF+CG+\zeta_3(-AF-BG+CF+CG)) \tag{5.23}$$

where F and G are given in (5.1). Finally, one can similarly show that the $k=2$ term in the second line of (5.15) is

$$3\frac{J_1J_6J_{108}}{J_2J_3^2}(-AG+BF+BG-CF+\zeta_3(AF+BG-CF-CG)). \tag{5.24}$$

Combining (5.22)–(5.24) and simplifying, the second line of (5.15) equals

$$-q\frac{J_2^3J_3^2J_6^2J_{24}J_{36}}{J_1^2J_4^2J_8J_{12}^2J_{18}^2}\left(q^4\frac{J_3^3J_{18}J_{36}^2J_{216}}{J_6^3J_9J_{72}J_{108}}+2\frac{J_1J_6J_{108}}{J_2J_3^2}\left(-2AG+2BF+BG-AF-CF+CG\right)\right). \tag{5.25}$$

As (5.15) is the sum of (5.20) and (5.25), we obtain

$$\frac{J_2^3J_3^2J_6^2J_{36}}{J_1^4J_4^2J_{12}J_{18}^2}\left(\frac{J_2J_8J_{12}^2}{J_4J_6J_{24}}\left(\frac{J_3^3J_{18}^2J_{72}J_{108}^2}{J_6^3J_9J_{36}J_{54}J_{216}}-2q^2\frac{J_1J_6J_{108}}{J_2J_3^2}(2AD-AE-BD-BE-CD+2CE)\right)\right.$$
$$\left.+q\frac{J_4^2J_{24}}{J_8J_{12}}\left(q^4\frac{J_3^3J_{18}J_{36}^2J_{216}}{J_6^3J_9J_{72}J_{108}}+2\frac{J_1J_6J_{108}}{J_2J_3^2}\left(-2AG+2BF+BG-AF-CF+CG\right)\right)\right). \tag{5.26}$$

Note that (5.26) can be rearranged as

$$\begin{aligned}\frac{J_3^2J_6^2J_{36}}{J_{12}J_{18}^2}\Bigg(&\frac{J_3^3J_{12}^2J_{18}^2J_{72}J_{108}^2}{J_6^4J_9J_{24}J_{36}J_{54}J_{216}}\frac{J_2^4J_8}{J_1J_4^3}\frac{1}{J_1^3}\\&-2q^2\frac{J_{12}^2J_{108}}{J_6J_{24}}\left(2AD-AE-BD-BE-CD+2CE\right)\frac{J_2^4J_8}{J_1J_4^3}\frac{1}{J_1^3}\frac{J_1J_6}{J_2J_3^2}\\&+q^5\frac{J_3^3J_{18}J_{24}J_{36}^2J_{216}}{J_6^3J_9J_{12}J_{72}J_{108}}\frac{J_2^3}{J_1J_8}\frac{1}{J_1^3}\\&+2q\frac{J_{24}J_{108}}{J_{12}}\left(-2AG+2BF+BG-AF-CF+CG\right)\frac{J_2^3}{J_1J_8}\frac{1}{J_1^3}\frac{J_1J_6}{J_2J_3^2}\Bigg).\end{aligned} \tag{5.27}$$

Moreover, one can check

$$4\Psi_2^3(q^9,-1,-1;q^{18})-2\Psi_1^3(q^9,-1,-1;q^{18})$$
$$=-\frac{3}{2}q^{-9}\frac{J_{18}J_{27}J_{108}J_{162}^5}{J_{36}^2J_{54}J_{81}J_{324}^3}\left(\frac{j(q^{27};q^{162})}{j(-q^{27};q^{162})}+\frac{j(q^{81};q^{162})}{j(-q^{81};q^{162})}\right). \tag{5.28}$$

We now insert (2.33)–(2.37) into (5.27) and (2.38) into (5.10) and combine with (5.28). After substituting the resulting expressions into (5.6) and recalling (5.2) and (5.3), we arrive at (5.4). This proves the result. □

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