NONCOMMUTATIVE REAL ALGEBRA

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Modern real algebra arguably started with Artin's solution of Hilbert's 17th problem in the late 1920s, for which the theory of ordered fields was introduced. However, a lot of work had already been done in the 19th century in the special case of the field of real numbers, with numerous results about the real zeroes of polynomials, and also with Sylvester's inertia theorem which established one of the first links between quadratic forms and real algebra.

Shortly thereafter, valuations were thrown into the mix, and the triad quadratic forms–orderings–valuations has played an important role ever since in the development of real algebra. In this context, results of note include the Baer–Krull theorem and Pfister's local-global principle.

The theory of orderings was later extended to commutative rings, opening up real algebra and real algebraic geometry to many new domains (ranging from classical algebra, to geometry and analysis), with new tools and new results. Here, the discovery of the real spectrum by Coste and Roy in the late 1970s was a game-changer. In the early 1980s, Mahé again provided a strong connection between real algebra/geometry and quadratic forms by showing that the connected components of the real spectrum of a commutative ring A (and in particular the connected components of a real variety) are in bijective correspondence with the signatures on A, i.e., the ring homomorphisms from the Witt ring (of classes of nonsingular quadratic forms) of A to the integers.

A bit more than a decade ago, Astier and I started investigating the noncommutative direction. We have been developing the real algebraic aspects of central simple algebras with involution over fields, and Azumaya algebras with involution over commutative rings, using the theory of hermitian forms over these algebras as our central tool. In my talk, I will report on our work, keeping the technical level to a minimum.